Divisor Electoral Apportionment Method
Based on
Atkinson Social Welfare Function

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Abstract

A divisor method for apportionment with the threshold of the Stolarsky mean can be derived by maximizing the Atkinson Social Welfare Function and minimizing the Generalized Entropy Index. We evaluate these divisor electoral apportionment methods on a unique parameter, or unique axis, of the Stolarsky mean. Though we would hesitate to choose one Social Welfare Function, we recommend the divisor method using the threshold of the logarithmic mean. Theorem 5.1 from Balinski-Young (1982) shows that on the unique axis of the Stolarsky mean, the divisor method using a logarithmic mean is the most unbiased method by which to apportion seats.

Keywords

Apportionment; Atkinson Social Welfare Function; divisor method; Stolarsky mean; Generalized Entropy Index.

JEL classification codes

D63, D72

1. Introduction

Balinski and Young (1982) showed in their theorem 4.3 that divisor methods are the only apportionment methods that avoid population paradoxes (as well as the Alabama and new states paradoxes). They also reported that the Webster method (Sainte-Lague method) is the only unbiased divisor method (Appendix A5). However, the Webster method cannot ensure a positive number of seats to every state. Table 1 shows the case of Canada, and Table 2 shows the case of the Upper House of Japan.

Wada (2010) used the Nash Social Welfare Function as the basis for evaluating the one-person-one-vote problem and created a decomposable index[1] to break down the single problem into two: the problem of apportionment and that of districting. If we minimize the decomposable index of the apportionment part, we can obtain the divisor method using a logarithmic mean, which ensures at least one seat to every state.

In this paper, we derive a divisor method for apportionment with the thresholds of the Stolarsky mean, which includes the U.S. House of Representatives, Sainte-Lague, and d'Hondt methods. We maximize the Atkinson Social Welfare Function—which includes the Rawlsian, Nash, and Benthamian Social Welfare Functions—and minimize the Generalized Entropy Index, which includes the Mean Log Deviation, the Theil Index (Kullback–Leibler divergence, relative entropy), and the Coefficient of Variance.

Then we can evaluate these divisor electoral apportionment methods on a unique parameter or a unique axis.[2] Though we would hesitate to choose one Social Welfare Function—that is, one Generalized Entropy Index—we could recommend the divisor method using a logarithmic mean, which is derived by maximizing the Nash Social Welfare Function—that is, by minimizing the Mean Log Deviation. By using theorem 5.1 in Balinski-Young (1982) we show that, on the axis of the Stolarsky mean,

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[1] This index resembles the Theil index, but in our more general framework, it would be better to consider it as Mean Log Deviation. See footnotes 4 and 5.

[2] Lauwers and Puyenbroeck (2008) used the Stolarsky mean of two parameters and induced all of Huntington's five traditional divisor methods, including the Dean method (harmonic mean).

the divisor method using a logarithmic mean is the most unbiased divisor apportionment method to ensure that at least one seat is assigned to each state.

2. Atkinson Social Welfare Function and Generalized Entropy Indexes

With the utilitarian principle, the Atkinson Social Welfare Function for a society of *N* persons is defined as follows.

$$ASWF^{\varepsilon}\left(\mathbf{y}\right)=\sum_{i=1}^{N}\frac{1}{\left(1-\varepsilon\right)}\left(y_i^{(1-\varepsilon)}-1\right),$$

where y_i is the wealth of *i* in the society of *N* persons, and ε is the coefficient of the relative risk aversion. When $\varepsilon\to1$, the function becomes the natural log of the Nash product or Nash Social Welfare Function.

$$ASWF^{1}\left(\mathbf{y}\right)=\sum_{i=1}^{N}\ln\left(y_i\right)=\ln\left(\prod_{i=1}^{N}y_i\right)$$

We can redefine the Atkinson Social Welfare Function in the case of $\varepsilon\to\infty$ as $ASWF^{\infty}\left(\mathbf{y}\right)=\min_i y_i$ (the Rawlsian Social Welfare Function), in the case of $\varepsilon\to1$ as $ASWF^{1}\left(\mathbf{y}\right)=\prod_{i=1}^{N}y_i$ (the Nash Social Welfare Function), and in the case of $\varepsilon=0$ as $ASWF^{0}\left(\mathbf{y}\right)=\sum_{i=1}^{N}y_i$ (the Benthamian Social Welfare Function). We could also imagine the case of $\varepsilon\to-\infty$, hence arriving at $ASWF^{-\infty}\left(\mathbf{y}\right)=\max_i y_i$.

If we use relative wealth (that is, if we divide each individual’s wealth by the average wealth) and divide the total social welfare by the total population *N*, we get an index of equity. (We will multiply the index by $\left(-1/\varepsilon\right)$ for mathematical convenience.) If we redefine $\alpha\equiv1-\varepsilon$, we see that this index represents the Generalized Entropy Index.

$$E^{\alpha}\left(\mathbf{y}\right)=\frac{1}{\alpha\left(\alpha-1\right)}\left(\sum_{i=1}^{N}\frac{1}{N}\left(\left(\frac{y_i}{\bar{y}}\right)^{\alpha}-1\right)\right)$$

From the special values of the parameter, α (that is, ε), we get some familiar indexes.

$\alpha \to -\infty$ ($\varepsilon \to \infty$, corresponding to the Rawlsian)

$$E^{-\infty}(\mathbf{y}) = -\min_i y_i$$

$\alpha \to 0$ ($\varepsilon \to 1$, corresponding to the Nash)

$$E^{0}(\mathbf{y}) = \frac{1}{N}\sum_{i=1}^{N}\log\left(\frac{\bar{y}}{y_i}\right) \qquad \text{Mean Log Deviation}$$

$\alpha \to 1$ ($\varepsilon \to 0$, corresponding to the Benthamian)

$$E^{1}(\mathbf{y}) = \frac{1}{Y}\sum_{i=1}^{N} y_i \log\left(\frac{y_i}{\bar{y}}\right) \qquad \text{Theil Index}$$

$\alpha = 2$ ($\varepsilon = -1$)

$$E^{2}(\mathbf{y}) == \frac{1}{2}\left(\left(\frac{1}{N}\sum_{i=1}^{N}\left(y_i - \bar{y}\right)^2\right)^{\frac{1}{2}} \Big/ \bar{y}\right)^2 = \frac{1}{2}CV^2$$

Half of the squared Coefficient of Variation[3] $\alpha \to \infty$ ($\varepsilon \to -\infty$),

$$E^{\infty}(\mathbf{y}) = -\max_i y_i$$

3. Apportionment Method

As shown in section 2, maximizing Atkinson Social Welfare ($ASWF^{1-\alpha}$ or $ASWF^{\varepsilon}$) means minimizing the corresponding Generalized Entropy Index (E^{α} or $E^{1-\varepsilon}$). Wada (2010) obtained a divisor apportionment method with a logarithmic mean from the Nash Social Welfare function or, in his case, the Theil Index. Here, we derive divisor apportionment methods from the more general Atkinson Social Welfare Function or Generalized Entropy Index.

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[3] $\alpha = -1$($\varepsilon = 2$) would be considered as half of the squared coefficient of variation of a person supported by one dollar.

$$E^{-1}(\mathbf{y}) = \frac{1}{2}\left(\left(\frac{1}{Y}\sum_{i=1}^{N} y_i\left(\frac{1}{y_i} - \frac{1}{\bar{y}}\right)^2\right)^{\frac{1}{2}} \Big/ \frac{1}{\bar{y}}\right)^2 = \frac{1}{2}cv^2$$

Let us suppose that the population of state *j* is N_j and that the number of representatives is n_j; the total population is *N*, and the total number of representatives is *n*. If we assume that the representatives are wealth and equally divided in state *j*, the number of representatives per person can be expressed as n_j / N_j, and the Generalized Entropy Index for *N* person in *k* states is as follows.

$$E^{\alpha} = \frac{1}{\alpha(\alpha-1)}\left(\left(\sum_{j=1}^{k}\left(\frac{N_j}{N}\right)^{1-\alpha}\left(\frac{n_j}{n}\right)^{\alpha}\right)-1\right)$$

$\alpha \to 0$ Mean Log Deviation ($\varepsilon \to 1$ (corresponds to the Nash))[4]

$$E^{0} = \frac{1}{N}\sum_{j=1}^{k} N_j \log\left(\frac{n/N}{n_j/N_j}\right).$$

$\alpha \to 1$ Theil Index ($\varepsilon \to 0$ (corresponds to the Benthamian))[5]

$$E^{1} = \frac{1}{n}\sum_{i=1}^{k} N_j \frac{n_j}{N_j}\log\left(\frac{n_j/N_j}{n/N}\right)$$

Thus, from the viewpoint of maximizing the Atkinson Social Welfare ($ASWF^{1-\alpha}$ or $ASWF^{\varepsilon}$) of *N* persons in *k* states—that is, minimizing the corresponding Generalized Entropy Index (E^{α} or $E^{1-\varepsilon}$)—we derive the optimal apportionment as follows.

When $\alpha \neq 0$ and $\alpha \neq 1$ ($\varepsilon \neq 1$ and $\varepsilon \neq 0$), the optimal apportionment must satisfy the following condition:

$\forall s,t$, $s \neq t$ and $n_s > 0$, $n_t \geq 0$

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[4] Wada (2010) transformed the equation as follows and called it the Theil index. Because this is an identity, we get the same divisor method (one with a logarithmic mean.)

$$\frac{1}{N}\sum_{j=1}^{k} n_j \frac{N_j}{n_j}\log\left(\frac{N_j/n_j}{N/n}\right)$$

[5] Wada (2010) transformed the equation as follows and called it the Mean Log Deviation. Because this is an identity, we get the same divisor method (one with an identric mean.)

$$\frac{1}{n}\sum_{i=1}^{k} n_j \log\left(\frac{N/n}{N_j/n_j}\right)$$

$$\begin{aligned}&\frac{1}{\alpha(\alpha-1)}\left(N_s^{1-\alpha}(n_s-1)^{\alpha}+N_t^{1-\alpha}(n_t+1)^{\alpha}\right)\\&\qquad\geq\frac{1}{\alpha(\alpha-1)}\left(N_s^{1-\alpha}n_s^{\alpha}+N_t^{1-\alpha}n_t^{\alpha}\right)\end{aligned}$$

The term “optimal” implies the following:

$$\min_{n_s>0}\left(\frac{N_s}{\left(\frac{n_s^{\alpha}-(n_s-1)^{\alpha}}{\alpha}\right)^{-\frac{1}{1-\alpha}}}\right)\geq\max_{n_t\geq 0}\left(\frac{N_t}{\left(\frac{(n_t+1)^{\alpha}-n_t^{\alpha}}{\alpha}\right)^{-\frac{1}{1-\alpha}}}\right)$$

Because this apportionment satisfies the divisor methods (Balinski and Young 1982), we can restate it as follows: Find a divisor *x* so that n_j s, which are the “special rounded” numbers of the quotients of states, N_j/x, add up to the required total, *n*. Here, “special rounded” means rounded up when the quotient is equal to or bigger than the Stolarsky mean, instead of the arithmetic mean, of both side integers ((n_j-1) and n_j).

In brief, we get the following proposition.

Proposition 1

For maximizing the Atkinson Social Welfare Function—that is, minimizing the Generalized Entropy Index—we must use the divisor apportionment method with the threshold of the Stolarsky mean of both side integers.

When $\alpha\to 0$, which is the case of Mean Log Deviation and which corresponds to the Nash Social Welfare Function, the term “optimal” implies the following, and the threshold becomes the logarithmic mean:[6]

$$\min_{n_s>0}\left(\frac{N_s}{\frac{n_s-(n_s-1)}{\log n_s-\log(n_s-1)}}\right)\geq\max_{n_t\geq 0}\left(\frac{N_t}{\frac{(n_t+1)-n_t}{\log(n_t+1)-_t\log n_t}}\right).$$

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[6] We can consider $0^0=1$.

When $\alpha \to 1$, which is the case with the Theil Index and which corresponds to the Benthamian Social Welfare Function, the term “optimal” implies the following and the threshold becomes the identric mean:[7]

$$\min_{n_s>0}\left(\frac{N_S}{(n_s)^{n_s}\big/e(n_s-1)^{(n_s-1)}}\right) \geq \max_{n_t\geq 0}\left(\frac{N_t}{(n_t+1)^{(n_t+1)}\big/e(n_t)^{n_t}}\right)$$

When $\alpha \to \infty$, we arrive at the round-up number for the threshold. It represents the Jefferson method (d’Hondt method). When $\alpha \to -\infty$, which corresponds to the Rawlsian Social Welfare Function, we get the round-down. It represents the Adams method (1+d’Hondt method). $\alpha = -1$ ($\varepsilon = 2$, in the case of a risk averter) gives the geometric mean and the Hill method (U.S. House of Representatives method). $\alpha = 2$ ($\varepsilon = -1$, in the case of a risk lover) gives the arithmetic mean and the Webster method (Sainte-Lague method). $\alpha = 1/2$ ($\varepsilon = 1/2$, in the case of a risk averter) gives a power mean with exponent = 1/2. We collect the results in Table 3.

All thresholds of the divisor methods induced from the Generalized Entropy Index—that is, the Atkinson Social Welfare Function—are the Stolarsky mean of both side integers ($(n_j - 1)$ and n_j) with the unique parameter α. According to theorem 5.1 in Balinski-Young (1982), we can say that the divisor method with a larger α (smaller ε) favors large states and that with a smaller α (larger ε) favors small states. As theorem 5.3 in Balinski-Young (1982) shows, the unbiased divisor method is the Webster method, which is the case of $\alpha = 2$ (arithmetic mean). Since the arithmetic mean (threshold) of 0 and 1 is 0.5, it cannot ensure a positive number of seats to every state. For making the Stolarsky mean between 0 and 1 equal 0 or ensuring positive seats, α must not be positive. The limit is the case of $\alpha \to 0$, which is the case of Mean Log Deviation or the Nash Social Welfare Function.

Proposition 2

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[7] We can consider $0 \log 0 = 0$, and $0^0 = 1$.

In the divisor method using the Stolarsky mean, the divisor method using a logarithmic mean is the most unbiased divisor apportionment method to ensure that at least one seat is assigned to each state.

4. Apportionment from the Viewpoint of Equity between Representatives

From the viewpoint of the representatives, we could consider the following cost function.

$$c_i(x_i) = \frac{1}{(1-\delta)}\left(x_i^{(1-\delta)} - 1\right),$$

where x_i is the population that representative i represents. Here, parameter δ indicates the following information.

$\delta < 0$ increasing marginal cost

$\delta = 0$ constant marginal cost

$\delta > 0$ decreasing marginal cost

Let us create a social cost function of n representatives.

$$scf^{\delta} = \sum_{i=1}^{n} c_i(x_i) = \sum_{i=1}^{n} \frac{1}{(1-\delta)}\left(x_i^{(1-\delta)} - 1\right).$$

If we use the relative cost (that is, if we divide each representative's cost by the average cost) and divide the total social cost by the total number of representatives, we get an index of equity between the representatives. (We will multiply the index by $\left(-1/\delta\right)$ for mathematical convenience.) If we redefine $\beta \equiv 1-\delta$, it becomes clear that this index represents the Generalized Entropy Index.

$$e^{\beta} = \frac{1}{\beta(\beta-1)}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\bar{x}}\right)^{\beta}\right) - 1\right)$$

For the case of k states, we can rewrite it as follows.

$$e^{\beta} = \frac{1}{\beta(\beta-1)}\left(\left(\sum_{j=1}^{k}\left(\frac{N_j}{N}\right)^{\beta}\left(\frac{n_j}{n}\right)^{1-\beta}\right) - 1\right)$$

We then obtain the following divisor method. (See also Table 3.)

$\beta \to \infty$ ($\delta \to -\infty$ (increasing marginal cost))

Divisor method with round-down (Adams method (1+d'Hondt))

$\beta = 2$ ($\delta = -1$ (increasing marginal cost))

Divisor method with geometric mean (Hill method (the U.S. House of Representatives))

$\beta \to 1$ ($\delta \to 0$ (constant marginal cost))

Divisor method with logarithmic mean

$\beta = 1/2$ ($\delta = 1/2$ (decreasing marginal cost))

Divisor method with powered mean with exponent = 1/2

$\beta \to 0$ ($\delta \to 1$ (decreasing marginal cost))

Divisor method with identric mean

$\beta = -1$ ($\delta = 2$ (decreasing marginal cost))

Divisor method with arithmetic mean (Webster method (Saint Lague))

$\beta \to -\infty$ ($\delta \to \infty$ (decreasing marginal cost))

Divisor method with round-up (Jefferson method (d'Hondt))

5. Conclusion

What is the best apportionment method? To avoid population paradoxes, we must choose a divisor method. Balinski and Young (1982) recommended the Webster method because it is unbiased. However, as Tables 1 and 2 show, it cannot ensure a positive number of seats.

We recommend the apportionment method with the thresholds of the logarithmic mean. Every apportionment method induced from the Atkinson Social Welfare Function becomes a divisor method using a Stolarsky mean with a single parameter. A divisor apportionment method with a logarithmic mean is also the most unbiased of the methods that ensure at least one seat to every state.

From the viewpoint of the population, it is based on the risk averter and derived from the Nash Social Welfare Function. From the viewpoint of the representatives, it is based on a constant marginal cost case. If we consider postal cost, appointment time, etc., it would be natural for the marginal cost for a representative to be constant with the population.

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Table 1. Apportionment for Canada

	Population	Quota	Adams	Hill				Webster	Jefferson
			1+d'Hondt	U.S. House	$\mathbf{W}^A$		w^A	Saint Lague	d'Hondt
			minimum	geometric	**logarithmic**	powered(1/2)	*identric*	arithmetic	maximum
Canada (2001)	30007094	308	308	308	308	308	308	308	308
Ontario	11410046	117.12	114	117	117	117	117	117	119
Quebec	7237479	74.29	73	74	74	74	74	75	75
British Columbia	3907738	40.11	40	40	40	40	40	40	40
Alberta	2974807	30.53	30	31	31	31	31	31	31
Manitoba	1119583	11.49	12	11	11	11	12	12	11
Saskatchewan	978933	10.05	10	10	10	10	10	10	10
Nova Scotia	908007	9.32	10	9	9	9	9	9	9
New Brunswick	729498	7.49	8	7	7	7	8	8	7
Newfoundland and Labrador	512930	5.26	6	5	5	5	5	5	5
Prince Edward Island	135294	1.39	2	1	1	1	1	1	1
Northwest Territories	37360	0.38	1	1	1	1	1	0	0
Yukon (Territory)	28674	0.29	1	1	1	1	0	0	0
Nunavut (Territory)	26745	0.27	1	1	1	1	0	0	0

Table 2 Apportionment for Japanese Upper House

	Population	Quota	Adams	Hill				Webster	Jefferson
			1+d'Hondt	U.S.House	$\mathbf{W^A}$		w^A	Saint Lague	d'Hondt
			minimum	geometric	**logarithmic**	powered(1/2)	*identric*	arithmetic	maximum
Japan(2010)	128056026	73	73	73	73	73	73	73	73
Tokyo	13161751	7.50	6	6	6	6	7	7	9
Kanagawa	9049500	5.16	4	4	4	4	4	5	6
Osaka	8862896	5.05	4	4	4	4	4	5	6
Aichi	7408499	4.22	3	4	4	4	4	4	5
Saitama	7194957	4.10	3	4	4	4	4	4	5
Chiba	6217119	3.54	3	3	3	3	3	4	4
Hyogo	5589177	3.19	3	3	3	3	3	3	4
Hokkaido	5507456	3.14	3	3	3	3	3	3	4
Fukuoka	5072804	2.89	2	2	2	3	3	3	3
Shizuoka	3765044	2.15	2	2	2	2	2	2	2
Ibaraki	2968865	1.69	2	2	2	1	2	2	2
Hiroshima	2860769	1.63	2	1	1	1	1	2	2
Kyoto	2636704	1.50	2	1	1	1	1	2	2
Niigata	2374922	1.35	1	1	1	1	1	1	1
Miyagi	2347975	1.34	1	1	1	1	1	1	1
Nagano	2152736	1.23	1	1	1	1	1	1	1
Gifu	2081147	1.19	1	1	1	1	1	1	1
Fukushima	2028752	1.16	1	1	1	1	1	1	1
Gumma	2008170	1.14	1	1	1	1	1	1	1
Tochigi	2007014	1.14	1	1	1	1	1	1	1
Okayama	1944986	1.11	1	1	1	1	1	1	1
Mie	1854742	1.06	1	1	1	1	1	1	1
Kumamoto	1817410	1.04	1	1	1	1	1	1	1
Kagoshima	1706428	0.97	1	1	1	1	1	1	1
Yamaguchi	1451372	0.83	1	1	1	1	1	1	1
Ehime	1430957	0.82	1	1	1	1	1	1	1
Nagasaki	1426594	0.81	1	1	1	1	1	1	1
Shiga	1410272	0.80	1	1	1	1	1	1	1
Nara	1399978	0.80	1	1	1	1	1	1	1
Okinawa	1392503	0.79	1	1	1	1	1	1	1
Aomori	1373164	0.78	1	1	1	1	1	1	1
Iwate	1330530	0.76	1	1	1	1	1	1	1
Oita	1196409	0.68	1	1	1	1	1	1	0
Ishikawa	1170040	0.67	1	1	1	1	1	1	0
Yamagata	1168789	0.67	1	1	1	1	1	1	0
Miyazaki	1135120	0.65	1	1	1	1	1	1	0
Toyama	1093365	0.62	1	1	1	1	1	1	0
Akita	1085878	0.62	1	1	1	1	1	1	0
Wakayama	1001261	0.57	1	1	1	1	1	1	0
Kagawa	995779	0.57	1	1	1	1	1	1	0
Yamanashi	862772	0.49	1	1	1	1	1	0	0
Saga	849709	0.48	1	1	1	1	1	0	0
Fukui	806470	0.46	1	1	1	1	1	0	0
Tokushima	785873	0.45	1	1	1	1	1	0	0
Kochi	764596	0.44	1	1	1	1	1	0	0
Shimane	716354	0.41	1	1	1	1	0	0	0
Tottori	588418	0.34	1	1	1	1	0	0	0

Table 3. Quotients needed for seats.

traditional name of the methods	Adams 1+d'Hondt	Hill U.S. House	**W^A**		*w^A*	Webster Sainte-Lague	Jefferson d'Hondt
Utility F.		risk averter			*neutral*	risk lover	
ε	∞	2	**1**	1/2	*0*	-1	$-\infty$
Atkinson SWF	Rawlsian SWF		**Nash SWF**		*Benthamian SWF*		
Generalized Entropy		$1/2cv^2$	**MLD**		*Theil index*	$1/2CV^2$	
α	$-\infty$	-1	**0**	1/2	*1*	2	∞
MC	increasing marginal cost		**constant**	decreasing marginal cost			
δ	$-\infty$	-1	**0**	1/2	*1*	2	∞
β	∞	2	**1**	1/2	*0*	-1	$-\infty$
Stolarsky mean	round-down minimum	geometric mean	**logarithmic mean**	powered mean (1/2)	*identric mean*	round arithmetic mean	round-up maximum
α	$-\infty$	-1	**0**	1/2	*1*	2	∞
Quotient needed for *n* seats							
1	0	0	**0**	0.2500	*0.3679*	0.5	1
2	1	1.4142	**1.4427**	1.4571	*1.4715*	1.5	2
3	2	2.4495	**2.4663**	2.4747	*2.4832*	2.5	3
4	3	3.4641	**3.4761**	3.4821	*3.4880*	3.5	4
5	4	4.4721	**4.4814**	4.4861	*4.4907*	4.5	5
6	5	5.4772	**5.4848**	5.4886	*5.4924*	5.5	6
7	6	6.4807	**6.4872**	6.4904	*6.4936*	6.5	7
8	7	7.4833	**7.4889**	7.4917	*7.4944*	7.5	8
9	8	8.4853	**8.4902**	8.4926	*8.4951*	8.5	9
10	9	9.4868	**9.4912**	9.4934	*9.4956*	9.5	10