Frequency and Severity of Some STV Paradoxes

Work in progress

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Abstract

Nonmonotonicity, the no-show paradox, and marginalization are major faults of STV/IRV. This paper extends my 2010 PCS paper, containing new results about the frequency of these faults, and comparing their frequency and their severity.

For STV elections with n voters and three-candidates, A, B, and C, the voters' rankings constitute a *profile*. The *triplet*, (a, b, c), of a profile counts candidates' first-place votes; we require each to be less than n/2. C denotes the eliminated candidate, and A the winner.

We obtain asymptotic values the proportions of nonmonotonic profiles as a function of triplets; these give lower bounds for individual profiles.

We compare frequencies for nonmonotonicity and no-show for a variety of profile types. For example, for competitive elections, ($c > (n+2)/4$), nonmonotonic profiles are more common than no-show profiles, but when $c < n/4$, none are nonmonotonic and many exhibit no-show.

Voters favoring B are marginalized -- STV ignores their second choices. Marginalization can be a problem when a majority of them rank C above A, i.e., when most who rank B first prefer C to A, as can occur in vote-splitting. Under STV, A is elected no matter how great this majority.

We conclude with observations about the occurrence of "later no harm" in STV.

Introduction

It has been shown that STV suffers substantially from both nonmonotonicity and from the no-show paradox. There has been some discussion about which is the greater fault. We discuss this question both from the standpoint of their frequency and their severity, and we extend that discussion to other significant problems with STV.

A profile for an STV election in which A is the winner *is upward nonmonotonic* or *violates upward monotonicity*, if the voters' preferences can be modified by only moving A up in the rankings so that A is no longer the winner. The *add no-show paradox* applies to a profile for an STV election if there is a set of voters with identical preference such that, if they are added to the profile, their vote changes the winner to a candidate they rank lower.

We begin with propositions about the proportion of profiles in STV that are nonmonotonic as a function of their triplets. The proofs are in the appendix. We compare the frequency of occurrences of nonmonotonicity and the no-show paradox as a function of some properties of profiles, and compare the severity of these two properties.

We then discuss marginalization and how it relates to nonmonotonicity. We conclude with a brief discussion of other problems with STV.

Generic notation and hypotheses: The propositions in this paper, unless otherwise stated, concern elections with three candidates A, B, and C, using STV. A *profile* is a set of rankings of the candidates by the voters. In a profile, the numbers of first-place votes for the candidate are a, b, and c, respectively, where a+b+c = n, the number of voters. We use the *triplet* (a, b, c) to describe the first-place vote totals of the candidates. We denote C as the candidate who is eliminated, and A as the winner. This implies a>c, and b>c--we ignore ties. We limit our discussion to profiles where each candidate gets fewer than n/2 first-place votes. Given a triplet (a, b, c), many of our propositions deal with examining the set of all profiles for which have the given triplet as its first-place vote totals. A useful concept (suggested by Joseph Ornstein) is the *competitive ratio*, c/a.

Propositions about nonmonotonicity

Proposition 1: Given a triplet (a, b, c), a necessary and sufficient condition for the existence of a profile with this triplet that violates upward monotonicity is $c \geq (n+3)/4$

Corollary: For a profile, P, with triplet (a, b, c) where $c \geq (n+3)/4$, P violates upward monotonicity if and only if P contains more than n/2-c BCA rankings.

We define a three-candidate profile to be *competitive* if $c \geq (n+3)/4$; its triplet is also defined to be *competitive.* Under this definition, one candidate can get almost half the first-place votes. A profile is *strongly competitive* if no candidate gets more than 40% of the first place votes; its triplet is also *strongly competitiv.*

Proposition 2: The proportion of all profiles with triplet (a, b, c) that are upward nonmonotonic is less than or equal to the proportions for all such profiles with triplets (a-1, b, c+1) and (a, b-1, c+1), provided c+1 is the smallest component; equality holds only if all proportions are 0 The proportion approaches ½ as c approaches 1/3.

Corollary: For fixed n, the proportion increases as a function of the competitive ratio c/a.

From Proposition 2 we see that for competitive profiles, those with minimal c have the smallest proportions of upward nonmonotonicity. Proposition 3 tell us that for strongly competitive profiles these proportions are all at least 2/7,while Proposition 4 shows that these smallest proportions are nonincreasing as n increases.

Proposition 3: For strongly competitive profiles with minimum c and maximum a, the proportion of upward nonmonotonic profiles is at least 2/7, equality holding if and only if n is congruent to 0 mod 20 and n >20. (Proof by looking at all congruence classes mod 20.)

Proposition 4: For strongly competitive profiles with minimum c and maximum a, and where n = 20v + u, 0 ≤ u ≤ 19, the proportion of upward nonmonotonic profiles is a decreasing function of v (except when u = 0, when it is constant) that approaches 2/7 as a limit.

We formulate the propositions about the asymptotic proportions of upward nonmonotonicity for all profiles with triplet (a, b, c), where c is a minimum. In order to avoid fractions, for very large n it is convenient to let n = 20v. For sufficiently large that in our calculations, we can replace v by v +/- u where |u| ≤ 10 makes no significant difference. Asymptotically, then, the value of c is 5v and the maximum value of a is 10v. Keeping c at its minimum value, 5v, a ranges between 10 and 5. Our generic triplet is then ((10-t)v, (5+t)v, 5v).

In the generic triplet, when a is large, the proportion of profiles for which A is the winner is large, but the proportion of nonmonotonic profiles is small. When a is small, and hence b is large, the proportion of profiles for which A is the winner is small while the proportion that are nonmonotonic is large. The proportion of all profiles with minimum c is the average of the upward nonmonotonic proportions for each triplet, weighted by the proportion for which A is the winner. Propositions 5, 6, and 8 determine the asymptotic averages for, that are, respectively, competitive, strongly competitive, and those for which c ≥ 3n/10.

Proposition 5: For all competitive triplets (a, b, c) with minimum c, the asymptotic proportion of profiles that are upward nonmonotonic is

$$\int_0^5 t(5-t)(10-t)dt \Big/ \int_0^5 (5+t)(5-t)(10-t)dt$$

This is 156.25/677.083333, which is approximately .2308.

Proposition 6: For all strongly competitive triplets (a, b, c) with minimum c, the asymptotic proportion of profiles that are upward nonmonotonic is

$$\int_2^3 t(5-t)(10-t)dt \Big/ \int_2^3 (5+t)(5-t)(10-t)dt$$

This is 46.25/140.083333, which is approximately .3302.

Proposition 7: If c >= .3n, then all competitive profiles are strongly competitive.

Proposition 8: For all competitive triplets (a, b, c) with c = 3n/10, the asymptotic proportion of profiles that are upward nonmonotonic is

$$\int_0^1 (1+t)(2-t)(4-t)dt \Big/ \int_0^1 (3+t)(2-t)(4-t)dt$$

This is 7.46667/18.28, which is approximately .4064.

These rather large proportions of upward nonmonotonic profiles are for the smallest value of c in their domains, and hence are lower bounds for the proportions with larger values of c. When c is very close to n/3, the proportions increase toward ½, but in these profiles, chance events, and errors might be factors in determining the winner in an actual election, as has been noted in recounts of very close elections under plurality. However, several recent examples of strongly competitive elections that are not so close (mayoral elections in Burlington, VT, Aspen, CO, Oakland, CA, for example), show that such profiles can arise in real elections.

Examining all profiles with certain properties isn't the same as examining actual elections. In fact, profiles that exhibit upward nonmonotonicity frequently contain majority cyclic triples. While such triples might occur rarely, we would not expect them to occur often. It is not difficult to calculate the asymptotic proportions of upward nonmonotonic profiles that do not contain majority cyclic triples, as shown in Propositions 10-12.

Definition: A profile is Condorcet if it has a Condorcet leader, i. e., a candidate that beats all others in pairwise comparison.

Ignoring ties, profiles that are not Condorcet have majority cyclic triples.

Proposition 9: For all triplets (a, b, c) with minimum c, the asymptotic proportion of Condorcet strongly competitive profiles that are upward nonmonotonic is int((a+c-b+1)/2) /(a+1).

Corollary: For given n, this proportion increases as a increases, as c increases, and as the competitive ratio c /a increases; it approaches ½ as c approaches b.

Proposition 10: For all competitive triplets (a, b, c) with minimum c, the asymptotic proportion of Condorcet profiles that are upward nonmonotonic is

$$\int_0^5 t(5-t)^2 dt \Big/ \int_0^5 (5+t)(5-t)(10-t)dt$$

This is 52.083333/677.083333, which is approximately .1102.

Proposition 11: For all strongly competitive triplets (a, b, c) with minimum c, the asymptotic proportion of Condorcet profiles that are upward nonmonotonic is

$$\int_2^3 t(5-t)^2 dt \Big/ \int_2^3 (5+t)(5-t)(10-t)dt$$

This is 15.43333/140.083333, which is approximately .1102.

Proposition 12: For all competitive triplets (a, b, c) with c = 3n/10, the asymptotic proportion of Condorcet competitive profiles that are upward nonmonotonic is

$$\int_0^1 (1-t)(2-t)^2 dt \Big/ \int_0^1 (3+t)(2-t)(4-t)dt$$

This is 3.25/18.25, which is approximately .1781.

Joseph Ornstein (Ornstein, 2010). used another approach to explore the likelihood of nonmonotonic STV elections. He created a two-dimensional geometric model simulating three-candidate elections using two independent criteria to form a grid on which voters and candidates can be placed. He used a variety of positions for the candidates, and for each he used a variety of positions of the voters. Each voter ranked the candidates according to their distance from the voter's position. He found proportions of upward nonmomotonic profiles consistent with the results above, but with very few majority cyclic triples

Nonmonotonicitiy and no-show

For a profile to be subject to the no-show paradox (Fishburn and Brams, 1983), it is necessary that b > a. By contrast, for a profile to be upward nonmonotonic it is necessary that c > (n+2)/4. These describe quite different sets of profiles.

For a given n, the set of all profiles for which b > a contains a larger proportion subject to the no-show paradox than are upward nonmonotonic.

Also for a given n, a comparison of the two properties can be described in terms of the values of c. For competitive profiles, where c > (n+2)/4, the proportion of upward nonmonotonic exceeds that of no-show. If we consider only profiles where c ≤ (n+2)/4, there are no upward nonmonotonic profiles, but those showing the no-show paradox are not infrequent. If we look at all profiles where c > n/6 (a limiting value for down nonmonotonicity and drop no-show), add no-show occurs more often than upward nonmonotonicity. All in all, these two properties occur frequently. The preceding results are obtained from computer searching.

For the region of triplets that are feasible for both upward nonmonotonicity and add no-show, there are more profiles that exhibit no-show that nonmonotonicity.
However, as the competitive ratio decreases, the number of voters needed to produce no-show increases, decreasing the seriousness of the no-show property.

A fundamental principal of democracy is that government is by consent of the governed. Voting is an important part of that consent. Voters expect that voting should satisfy both monotonicity and participation. Upward nonmonotonicity and add no-show are violations of these two properties. Is it worse to vote and risk the possibility that by voting you cause a favorite candidate to lose, or find that by the act of voting you might cause your least-favored candidate to win? While people differ about which is worse, there is agreement that both are serious faults.

Marginalization

In an STV election with candidates A, B, and C, where C is eliminated and A is the winner, voters who rank B first are *marginalized*, in that it their second choices play no role in determining the winner. Marginalization is *significant* when a majority rank C above A, i.e., when most of those who rank B first in first place prefer C to A. Significant marginalization can occur in vote-splitting. If those who rank B first prefer C to A by a bigger margin than a-c, then A will win even though a majority of all voters prefer C to A. That doesn't necessarily imply that C should win, but it raises the question the question about a system that automatically declares A to be the winner.

Proposition M1: Given a profile in STV with three candidates, none of which has a majority of first-place votes. If the profile exhibits upward nonmonotonicity then after the elimination of one candidate, the losing candidate is significantly marginalized.

Unlike upward nonmonotonicity, significant marginalization can occur even
$c \leq (n+2)/4$, as shown in the following example.

Example: Consider the triplet (41, 35, 24)
As long as at least 10 voters rank the candidates CAB, A will win. This occurs in 15/25 of the triplets. Assuming A is the winner, then as long as 27 voters rank the candidates BCA, and this occurs in ¼ of the triplets, the profile violate upward monotonicity.

Later no harm

Richie and Bouricius (FairVote Archives) argue that we should consider the "Later No Harm" criterion when evaluating a voting system. They predict that under Approval Voting, voters will "bullet vote" (writing in their favorite alone) because voting for their second choice would reduce the chances that their favorite will win. But IRV is not immune to Later No Harm. Under some circumstances, voters who express their full preferences may cause their least favored candidate to win.

This possibility is illustrated by the following example with four candidates A, B, C, and D, where D is the least preferred candidate, and is the first eliminated under IRV. In this example, the two voters who rank D first may find all candidates except A as acceptable. But if they include their second-choice C on their ballot, look what happens with these preferences:

2 DCBA
3 CDBA
4 BADC
5 ABDC

If the two voters who prefer D include their second choice C in their ballots, then following the elimination of D, B is eliminated and A wins over C 9 to 5. However, if they bullet vote for D, then C is eliminated, and B wins over A 7 to 5. These D-voters prefer B to A, but by voting their full preference they cause B to be eliminated, so that their least-preferred candidate, A, wins. Thus they do harm by including more than their first choice on their ballot.

Under IRV, a voter who includes his second choice on the ballot will not decrease the likelihood of a win by his preferred candidate, but he may increase the chance that his least preferred candidate will win. Furthermore, such a voter would likely be unaware of the consequence of this vote. By contrast, under Approval Voting, a voter who casts a vote for his second choice can decrease the likelihood that his preferred candidate will win, but it also decreases the likelihood that his least preferred candidate will win. This may be a welcome opportunity, and the voter may choose which is more important to him. When his preferred candidate has little chance but his least preferred candidate has a good chance of winning, the voter may opt to include his second choice.

References

FairVote Archives, http://archive.fairvote.org/?page=2261

Fishburn, Peter, and Brams, Steven, Paradoxes of preferential voting, Mathematics Magazine, (1983), 56, 207-214

Norman, Robert Z,, The relationship between monotonicity failure and the
no-show paradox. PCS meeting March 2010.

Ornstein, Joseph, High Prevalence of Nonmonotonic Behavior in Simulated 3-Candidate
STV Elections, PCS meeting March 2010.

Appendix: Proofs of the propositions

This note document contains the proofs that I constructed in October for another purpose. They had different labels (letters rather than numbers). If you see Prop A, you can know that I missed a relabeling. Time is short and I won't be able to proofread tonight, nor to check on any updating

Prop. 1:
Proof of necessity: Suppose $c \leq (n+2)/4$. If A has a majority, then P is monotonic. If not, then after A is moved up in B-rankings to eliminate B, B will have at most $(n-2)/4$ first-place votes, and even if all of them are BCA rankings, C will not have a majority of first-place votes after B is eliminated. So P does not violate upward monotonicity.

Proof of sufficiency: The inequality $c \geq (n+3)/4$ assures hat no candidate has a majority. Then any profile in which $a>c$, $b>c$, with at least $c-1$ BCA rankings will violate monotonicty: Since

$b > c$ we can move B up to first place in all but $c-1$ B-rankings, eliminating B, and giving C a majority of $2c - 1 \geq (n+3)/2 - 1 = (n+1)/2$ votes.

Prop. 2: The proportion of all profiles with triplet (a, b, c) that are upward nonmonotonic is less than or equal to the proportions for all such profiles with triplets $(a-1, b, c+1)$ and $(a, b-1, c+1)$, provided $c+1$ is the smallest component; equality holds only if all proportions are 0 The proportion approaches ½ as c approaches 1/3.

Proof: Let $m = \text{int}((n+2)/2)$. This constitutes the smallest majority when there are n votes. The proportion of all profiles with triplet (a, b, c) that are upward nonmonotonic is $(n-m-a+1)/(b+1)$. If we increase c and decrease a by 1, n, m, and b remain unchanged so the proportion increases. If we increase c by 1 and decrease b by 1, the numerator is unchanged and the denominator decreases, so the proportion increases. We need only make sure that the new triplets conform to our requirement that the third entry be the smallest.)
[A note written in October: A possibly better proof is to use the alternate version of the proportion (see "prelimaries to proofs"] in which the proportion of all profiles with triplet (a, b, c) that are upward nonmonotnoic is $\text{int}((b+c-a)/2)/(b+1)$. From this it may be easier to see that the statement of the proposition is correct.]

Prop. 3: Let $n = 20v + u$, where $0 \leq u \leq 19$. The smallest value of c is $5v + \text{int}(u+6)/4)$. With a few exceptions, to be noted, $a = 8v + \text{int}(2u/5)$, and $b = n-a-c$.
A calculation of the proportions of upward nonmonotonic profiles for each congruence class mod 20 verifies the proposition.
There are no upward nonmonotonic profiles for $n = 22$, and when $n < 20$ there are upward nonmonotonic profiles only when $n = 17$.
The other exceptional cases are $n = 20, 23, 26$, and 30, each of which arises when $v = 1$. If we were to let $a = \text{int}(2n/5)$ and $c = \text{int}((n+6)/4)$, the result will be $b=c$, contrary to our assumptions that $b>c$. For each of these values of n there is exactly one triplet with upward nonmonotonic profiles, and the propositions can be verified directly.

Prop. 4: The calculation in the proof of Prop. 3 of the proportions of upward nonmonotonic profiles for each congruence class mod 20 produces a proof of Prop. D as well. The trickiness mentioned on the note I in the statement of Prop. D can be erased since it is taken care of in the proof just given.

Prop. 5: As noted in the paragraph preceding Proposition 5, we let $n = 20v$. To obtain asymptotic values we examine very large v. We assume v is large enough that replacing v by $v \pm u$ for $|u| < 11$ makes no significant difference in the following calculations. Asymptotically, then $c = 5v$ and a ranges from $10v$ to $5v$. We cover all possibilities by letting $a = (10-t)v$ and $b = (5+t)v$.

The following table for a profile, P, with triplet (a, b, c) shows how the proof develops:

N	a	b	c	prop	weight
20v	(10-t)v	(5+t)v	5v	t/(5+t)	((5-t)/5)(10-t)v(5+t)v

For A to win when C is eliminated, P must contain at least tv CAB rankings; hence the number of CBA rankings can be any integer from 0 to (5-t)v. The proportion of triplets ((10-t)v, (5+t)v, 5v) that satisfy this condition is (5-t)/5. This accounts for the factor (5-t)/5 in the “weight” column.

In order for C to win when A is moved up in some ranking with B in first place, the number of BCA rankings P must be at least 5v; hence the number of BAC ramkings can be any integer between 0 and tv. The proportion of these triplets satisfying this condition is t/(5+t), shown in “prop” column.

We want a weighted average of the values tv/(5+t), weighted by the proportion of triplets for each value of t for which A is the original winner and weighted by the product of the numbers a and b, namely (10-t)v*(5+t)v, to account for the fact that for fixed c, there are more profiles when a and b are nearly equal than when they differ.

This average is accomplished by the integrals stated in the proposition.

Prop. 6: The only difference between Prop. 6 and Prop. 5 is the restriction to strictly competitive profiles means that a and b can be no greater than 2n/5. That means that t ranges from 2 to 3, and the average stated in the proposition follow.

Prop. 7: The statement is obvious, and I would introduce it by saying so. It follows at once from a>c, b>c and a+b+c = n.

Prop. 8. The proof is very much like that for Prop. 5. Let n = 10v. Asymptotically, c = 3v. Then a = (4-t)v and b =(3+t)v, where 0 <= t <= 5, describe the possible values of a and b in a triple for a strongly competitive profile.

For A to win when C is eliminated, c1 > (t+1)v, so c-c1 can be any integer from 0 to (5-t)v. The proportion of triplets ((4-t)v, (3+t)v, 3v) that satisfy this condition is (2-t)/3.

In order for C to win when A is moved up in some B-rankings b2 must be at least 2v, so b-b2 can be any integer between 0 and (1+t)v. The proportion of triplets
((4-t)v, (3+t)v, 3v) that satisfy this condition is (1+t)v/(3+t).

We want a weighted average of the values (1+t)v/(3+t) weighted by the proportion of triplets for each value of t for which A is the original winner and weighted by the product of the numbers a and b, namely (4-t)v*(3+t)v.

This average is accomplished by the integrals stated in the proposition.

Prop. 9: For all triplets (a, b, c) with minimum c, the asymptotic proportion of Condorcet strongly competitive profiles that are upward nonmonotonic is
int((a+c-b+1)/2) /(a+1).

Proof: For C to be ranked above B it is necessary that c+a2 ≥ m, or equivalently, a+c-a1 ≥ m. This is equivalent to n-m-b ≥ a1 ≥ 0. There are n-m-b+1 such integers, so the proportion of profiles in which C is ranked above B is (n-m-b+1)/(a+1). This can also be formulated as int((a+c-b+1)/2)/(a+1).

Prop. 10: Proof is similar to that of Prop. 5. Let n = 20v. Asymptotically, c = 5v.
Then a = (10-t)v and b = (5+t)v, where $0 \leq t \leq 5$ describe the possible values of a and b in a triple for a competitive profile.
As in the proof of Prop. 5, the proportion of triplets ((10-t)v, (5+t)v, 5v) in which A wins when C is eliminated is (5-t)/5. And, the proportion of those triplets in which one can move A up in enough B-rankings for C to win when B is eliminated is t/(5+t).
For the profile to be Condorcet it cannot contain more than (5-t)v ABC rankings. So the proportion of those triplets that are Condorcet is (5-t)/(10-t).
The average is accomplished by the integrals stated in the proposition.

Prop. 11: As noted in the proof of Prop. 6, for profiles that are strongly competitive, $2 \leq t \leq 3$.
The result follows as in Prop. I.

Prop. 12: The proof essentially the same as the proof of Prop. 10, but using the information in the proof of Prop. 8. The only new information needed is that for the profile with triplet ((4-t)v, (3+t)v, 3v) to be Condorcet it cannot contain more than (2-t)v ABC rankings. So the proportion of these triplets that are Condorcet is (2-t)/4-t).