MONOTONICITY FAILURE UNDER IRV
WITH THREE CANDIDATES

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Abstract

A striking property of Instant Runoff Voting (IRV) is *monotonicity failure* — that is, getting more (first preference) votes can cause a candidate to lose an election and getting fewer votes can cause a candidate to win. Proponents of IRV have argued that monotonicity failure, while a mathematical possibility, is highly unlikely to occur in practice. The purpose of this note is to specify the precise conditions under which this phenomenon arises in the case of three candidates and then to apply these conditions to a 'random' set of simulated IRV elections, as well as to simulated election elections that meet various special conditions (e.g., single-peakedness) and to a large set of actual elections, in order to get a sense of the likelihood of monotonicity problems in varying circumstances.

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MONOTONICITY FAILURE UNDER IRV WITH THREE CANDIDATES

A striking feature of Instant Runoff Voting (IRV, also known as the Alternative Vote) is that getting more (first preference) votes can cause a candidate to lose an election and getting fewer votes can cause a candidate to win. Voting systems that are never vulnerable to this anomaly are said to be *monotonic*. Most voting systems, including Plurality Voting (First-Past-The-Post) are monotonic, but those that incorporate (actual or 'instant') runoffs may exhibit *monotonicity failure*.

1. IRV and Monotoncity Failure

On an IRV ballot, voters rank the candidates in order of preference. If one candidate has a majority of first preferences, that candidate is elected. Otherwise, the candidate with the fewest first preferences is eliminated and his or her ballots are transferred to other candidates on the basis of second preferences. This process is repeated until one candidate is supported by a majority of ballots and is elected. Here we consider only three-candidate contests, so IRV is limited to a single 'instant runoff' in the event none of three candidates is supported by a majority of first preferences. We also assume that all voters rank all three candidates.

Since monotonicity failure is a striking and counterintuitive phenomenon, it may be helpful first to provide a (more or less) real-world example — namely, a simplified version of the 2009 IRV election for mayor of Burlington, Vermont.[1] The Republican candidate was supported by 39% of the first preferences, the Democratic candidate by 27%, and the Progressive (left of Democrat) candidate by 34%. Thus the Democrat was eliminated, with his ballots transferring to one or other surviving candidate on the basis of Democratic second preferences, which were 37% for the Republican and 63% for the Progressive, representing 10% and 17% respectively of the total electorate. Thus in the instant runoff, the Republican got 39% + 10% = 49% and the Progressive won the election with 34% + 17% = 51%. Now consider a make-believe sequel. A third of the Republicans (13% of all voters) are so traumatized by the prospect of a Progressive mayor that they leave Burlington for more politically hospitable climes and are replaced by a like number of newcomers attracted by the prospect of a Progressive mayor. At the next election, all votes are cast exactly as before, except for the 13% of the electorate once made up of Republicans now replaced by Progressives. The Progressive candidate won a squeaker before, so with this new support surely he will win more comfortably this time. But in fact he does not win at all. The Republican candidate now has 26% of the vote, the Democrat 27%, and the Progressive 34%, so the instant runoff is now between the Democrat and the Progressive candidates, which the Democrat wins handily by gaining the second preferences of the (remaining) Republican voters (who find the prospect of a Democratic mayor at least marginally more tolerable than a Progressive one). So the consequence of the

[1] The description here is simplified in that there were other minor candidates, some voters cast 'truncated' ballots (that did not to rank all candidates), and voter preferences as expressed on the ballots were not entirely 'single-peaked' (i.e., not all Republicans ranked the Democrat over the Progressive and not all Progressives ranked the Democrat over the Republican).

Progressive candidate's first preferences being augmented by of 13% of the electorate is that he loses where before he won.

Let us describe the phenomenon more generally and a bit more precisely. Suppose we have three candidates *X*, *Y*, and *Z*, one of whom is to be elected under IRV. Suppose that *X* is the IRV winner and, more specifically, that *X* achieves this victory by getting into the runoff with *Y*, which *X* wins because *X* has sufficient second preference support from ballots that ranked *Z* first.

Suppose that some voters change their ballots by ranking *X* higher than they did before but no ballots change in any other way.[2] Suppose further that *X* thereby gains some additional first preference ballots but still not enough to win without a runoff. These additional first preference ballots must come at the expense of one or both other candidates. Suppose they come at the expense of *Y*, with the result that *Y* now has fewer first preference ballots than *Z*. Therefore, the runoff is now between *X* and *Z*, rather than between *X* and *Y*. Finally suppose that *Z* wins the runoff with *X*, because *Z* has disproportionate second preference support on the remaining ballots that rank *Y* first. (Clearly, *Z* would also have beaten *X* in a runoff given the original ballots but, lacking enough first preference votes, *Z* did not have the opportunity to do so because *Z* was eliminated after the first round.) We therefore have an instance of ('upward') monotonicity failure. Likewise, if X is *not* the IRV winner but would be if some voters change their ballots by ranking *X* lower than they did before while but no ballots change in any other way, we have an instance of ('downward') monotonicity failure.

Proponents of IRV have argued that monotonicity failure — while a mathematical possibility — is highly unlikely to occur in practice. The purpose of this note is to specify the precise conditions under which this phenomenon arises in the case of three candidates and then to apply these conditions to a 'random' set of simulated elections, as well as to simulated election elections that meet various special conditions (e.g., single-peakedness) and to a large set of actual elections, in order to get a sense of the likelihood of monotonicity problems in varying circumstances.

2. Preliminaries

An IRV *ballot profile* is a set of *n* rankings of the three candidates *X*, *Y*, and *Z*, where *n* is the number of voters. Given a particular ballot profile *B*, the candidate with the most first preferences is the *Plurality Winner*, the candidate with the second most first preferences is the *Plurality Runner-Up*, and the candidate with the fewest first preferences is the *Plurality Loser*. Let n (PW), n (P2), and n (PL) be the number of ballots that rank the Plurality Winner, the Plurality Runner-Up, and Plurality Loser first.[3] Given three candidates, it must be that $n\text{(PW)} > n/3$ and $n\text{(PL)} < n/3$. A Plurality

[2] In the make-believe Burlington sequel, the 13% of the voters whose ballots move the Progressive candidate from bottom to top rank presumably would also reverse the ranking of the Democratic and Republican candidates. However, even if (as stipulated above) the only change is move the Progressive from the bottom to the top, the same anomaly occurs.

[3] For simplicity, I ignore the possibility of ties, on the supposition that the number of voters is sufficiently large that ties almost never occur (and also because there is no standard way to break ties under IRV).

Winner who is the first preference of a majority of voters is a *Majority Winner*. The IRV winner is the Majority Winner if one exists and otherwise is either the Plurality Winner or the Plurality Runner-Up, depending on the outcome of the instant runoff between them.

Under a ballot profile B, the candidates have x, y, and z first preferences respectively, where $x + y + z = n$. Likewise x_y is the number of voters who have a first preference for X and second preference for Y (and therefore a third preference for Z), x_z is the number who have a first preference for X and a second preference for Z, so $x_y + x_z = x$; and likewise for other candidates. Under another ballot profile B', the candidates have x', y', and z' first preferences respectively.

If under ballot profile B a majority of voters rank X over Y, i.e. $x + z_x > y + z_y$, we say that 'X beats Y'. A *Condorcet Winner* is a candidate who beats both other candidates; a *Condorcet Loser* is a candidate who is beaten by both other candidates. If X beats Y, Y beats Z, and Z beats X or if Y beats X, X beats Z, and Z beats Y, we have a *Condorcet cycle*, and neither a Condorcet Winner nor a Condorcet Loser exists. A Majority Winner must be Condorcet Winner but, with three or more candidates, even the Plurality Winner may be a Condorcet Loser (e.g., the Republican candidate in the 2009 Burlington election) and the Plurality Loser may be a Condorcet Winner (e.g., the Democratic candidate in the same election). However, even if he is a Condorcet Winner, the Plurality Loser cannot be the IRV winner, because he is eliminated before it can get into a runoff (e.g., again the Democratic candidate).

Our aim is to specify the conditions under which an IRV ballot profile B is *vulnerable to monotonicity failure*, that is:

(i) the IRV winner is X but X would lose under some other ballot profile B' that differs from B only in that some voters rank X higher in B' than in B (*Upward Monotonicity Failure* or the *More-Is-Less Paradox*), or

(ii) X loses under IRV but X would win under some other ballot profile B' that differs from B only in that some voters rank X lower in B' than in B (*Downward Monotonicity Failure* or the *Less-Is-More Paradox*).

In either event, every voter ranks Y and Z the same way under both B and B'. Following Norman (2010), we refer to B and B' as *companion* profiles.

Let's round up some self-evident observations in the form of a lemma.

Lemma 1. In the event that ballot profile B' differs from B only in that some voters rank X higher in B' than in B, the following relationships hold:

(a) if X is a Majority Winner under B, X is also the Majority Winner under B';
(b) $x' \geq x$, $y' \leq y$, and $z' \leq z$ (in words, X is ranked first on no fewer, and Y and Z on no more, ballots under B than B');
(c) if X beats Y under B, X beats Y under B' (perhaps by larger margins);
(d) if X beats Z under B, X beats Z under B' (perhaps by larger margins);
(e) Y beats Z under B' if and only if Y beats Z under B (by exactly the same margin); and
(f) if Z beats Y under B' if and only if Z beats Y under B (by exactly the same margin).

In summary, despite being stronger than other candidates under B' as opposed to B in some such respects and weaker in none of them, X may fail to be the IRV winner under B' even though X is the IRV winner under B.

Lemma 2. If $n(\text{PL}) > n/4$, $3n/8 < n(\text{PW}) < n/2$ and there is no Majority Winner.

If $n(\text{PL}) > n/4$, $n(\text{PW}) + n(\text{P2}) < 3n/4$, so $n(\text{PW}) > .5(3n/4)$, and $n(\text{P2}) > n(\text{PL}) > n/4$, so $n(\text{PW}) < n/2$.

Two types of conditions pertaining to a ballot profile B are necessary to make B vulnerable to monotonicity failure with respect to candidate X and companion profile B'.

(1) A *Type 1 condition* specifies conditions on profile B that are necessary and sufficient to produce an instant runoff between X and Z, instead of X and Y, or vice versa.

(2) A *Type 2 condition* specifies conditions on profile B that are necessary and sufficient to make X the winner of one of the two possible runoffs but not the other.

The conjunction of a Type 1 condition and a Type 2 condition is necessary and sufficient to make profile B vulnerable to monotonicity failure.

3. Upward Monotonicity Failure

Let X be the IRV winner and let Z be the Plurality Loser under ballot profile B (so, if a runoff is needed, Y is the candidate that X beats in the runoff). We make these observations:

(a) ballot changes that move X upwards from last to second place cannot change the IRV winner, because (i) if X is already the Majority Winner, it remains so, (ii) if X is not the Majority Winner under B, X is still paired with Y in the runoff (because no first preferences have changed) and (iii) X still beats Y in this runoff (perhaps by a larger margin than before); and likewise

(b) ballot changes that move X upwards from last or second place to first place on ballots that had Z in first place cannot change the IRV winner, because either (i) X becomes a Majority Winner and wins without a runoff or (ii) it remains true that X is paired with and defeats Y in the runoff (perhaps by a larger margin than before), and therefore

(c) the essential difference between an initial ballot profile B and a companion ballot profile B' that produces Upwards Monotonicity Failure is that X is ranked first on some ballots in B' on which Y was ranked first in B.

Proposition 1 (cf. Lepelley et.al., 1996, p. 136). A ballot profile B under which X is the IRV winner and Z is the Plurality Loser is vulnerable to Upward Monotonicity Failure if and only if:

Condition 1U: $z > n/4$; and
Condition 2U: $z + y_z > x + y_x$.

In words, the Plurality Loser must (i) be the first preference of more than a quarter of the voters and (ii) beat the IRV winner under profile B. Note that both conditions pertain, in different ways, to the strength of the Plurality Loser Z — the more first preferences Z has, the more likely Z

is to beat another candidate, other things equal — so a profile that meets one condition is relatively more likely to meet the other as well. However, both conditions are somewhat at odds with the stipulation that Z is the Plurality Loser and is some sense the 'weakest' candidate. The implication is that Z must be somewhat 'strong' but not too 'strong.'

If ballot profile *B* is vulnerable to Upward Monotonicity failure, the Type 1 condition stipulates that it must be possible for *X* to gain enough first preference ballots at *Y*'s expense to *X* so that two things are simultaneously true under the companion ballot profile B': (i) *X* is still not a Majority Winner, and (ii) *Y* becomes the Plurality Loser instead of *Z*. Thus it must be that

$$n/2 - x > y - z.$$

Substituting $(n - y - z)$ into this expression in place of x and simplifying gives

$$n/2 - n + y + z > y - z,$$

which further simplifies to Condition 1U. Thus $z > n/4$ is necessary and sufficient to put *Z*, rather than *Y*, in the runoff with *X*. under B'.[4]

The Type 2 condition requires that *Z* beat *X* in the runoff under B'. From Lemma 1c, *Z* must beat *X* under *B* (i.e., $z + y_z > x + y_x$), as stipulated by C2U. While it is obvious that this condition is necessary for Upward Monotonicity Failure, it needs to be shown that, in conjunction with C1U, C2U is also sufficient.

In the event that $y_x \geq y - z$, all the first preference ballots that *X* must gain at *Y*'s expense to make *Y* the Plurality Loser under B' can come from the y_x ballots that would in any case transfer to *X* in a hypothetical runoff with *Z* under *B*, so *Z* beats *X* by the same margin under B' as under *B*. If $y_x < y - z$, it is evidently more difficult for *Z* to beat *X* under B' than under *B* because, to the extent that $y - z$ exceeds y_x, *Z* loses and *X* gains $[(y - z) - y_x]$ transferred ballots from *Y* in the runoff. Therefore, it must be that

$$z + y_z - [(y - z) - y_x] > x + y_x + [(y - z) - y_x].$$

Suppose to the contrary that

$$z + y_z - [(y - z) - y_x] \leq x + y_x + [(y - z) - y_x].$$

Removing parentheses and rearranging terms, we get

$$3z \leq x + 2y - (y_x + y_z)$$
$$3z \leq x + y.$$

Substituting $(n - z)$ for $(x + y)$ and further simplifying, we get $z \leq n/4$, contradicting C1U.[5]

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[4] Since *Y* becomes the Plurality Loser under B' and support for *Z* in unchanged, *X* must be the Plurality Winner under B' even if it was not under *B* (see Corollary 1.1).

[5] That this does not mean that the Condition 1U is by itself sufficient for Upward Monotonicity Failure. Rather it means that Condition 1U is by itself sufficient to imply that, if *Z* beats *X* under *B*, *Z* also beats *X* under B'.

Several corollaries follow from Proposition 1. First, nothing in Proposition 1 stipulates or implies that X is the Plurality Winner or Plurality Runner-Up under B, so we have the following:

Corollary 1.1. If a ballot profile B is vulnerable to Upward Monotonicity Failure, the IRV winner under B may be either the Plurality Winner or Plurality Runner-Up.

By Lemma 2 X cannot be a Majority Winner under B, we have:

Corollary 1.2. A ballot profile B is vulnerable to Upward Monotonicity Failure only if $n/2 > n(\text{PW}) > n(\text{PL}) > n/4$.

Corollary 1.2 implies that a ballot profile is vulnerable to Upward Monotonicity Failure only if there is a relatively close contest with respect to first preferences among the three candidates — specifically, all three candidates must get between 25% and 50% of the first preference votes.

In addition, Lemma 2 in conjunction with Condition 1U implies that:

Corollary 1.3. A ballot profile B is vulnerable to Upward Monotonicity Failure only if $n(\text{P2}) < 3n/8 < n(\text{P2})$.

Condition 2U implies that a ballot profile B in which Z is the Plurality Loser and X is the IRV winner is vulnerable to Upwards Monotonicity Failure only if X beats Y and Z beats X, so more generally :

Corollary 1.4. A ballot profile B is vulnerable to Upward Monotonicity Failure only if (i) the Plurality Loser is the Condorcet Winner or (ii) there is a Condorcet cycle.

4. Downward Monotonicity Failure

Let Y be the IRV winner and Z be the Plurality Loser (so, if a runoff is needed, X is the candidate beaten by Y in the runoff) under ballot profile B . We make these observations:

(a) ballot changes that move X downwards from second to third place cannot change the IRV outcome, because (i) if Y is the Majority Winner, it remains so, or (ii) if Y is not the Majority Winner, X is still paired with Y in the runoff (because no first preferences have changed), and (iii) Y beats X in this runoff (perhaps by a larger margin than before); and likewise

(b) ballot changes that increase Y's first preferences by moving X downwards cannot make X the IRV outcome because (i) if Y is the Majority Winner, it remains so, or (ii) X will no longer make it into the runoff with Y or (iii) X is still paired with and beaten by Y in the runoff (perhaps by a larger margin), and therefore

(c) the essential difference between the initial ballot profile B and the revised ballot profile B' that may produce Downwards Monotonicity Failure is that X is ranked second and Z first on some ballots in B' on which X was ranked first in B.

Proposition 2. (Cf. Lepelley et al., p. 139) A ballot profile B under which Y is the IRV winner and Z is the Plurality Loser is vulnerable to Downwards Monotonicity Failure if and only if:

Condition 1D: (a) $y < n/3$ and (b) $x_z > y - z$; and
Condition 2D: $y + y_z < n/2$.

In words, (1a) the IRV winner must be ranked first on fewer than one-third of the ballots, (1b) the number of ballots that rank the candidate who loses the runoff (i.e., *X)* first and the Plurality Loser second must exceed the margin by which the IRV winner leads the Plurality Loser with respect to first preferences, and (2) the number of ballot that rank the IRV winner first preference plus the number that rank the IRV winner first and the Plurality Loser second must be less than half of all ballots.

Note that Conditions 1D(a), 1D(b), and 2D all pertain, in different ways, to the strength of the Plurality Runner-Up (and IRV winner) *Y* — the fewer first preferences *Y* has, the smaller *Y*'s first preference advantage over *Z*, other things equal — so a profile that meets one condition is relatively more likely to meet the others as well. Moreover, Condition 1D(b) in addition appears to be a relatively easy condition to meet under many circumstances.

If ballot profile *B* is vulnerable to Downward Monotonicity failure, the Type 1 condition stipulates that *Z* must be able to gain enough first preference ballots at *X*'s expense to make *Y* (rather than either *Z* or *X*) the Plurality Loser; that is, that

$$x - y > y - z.$$

Rearranging and substituting $n - y$ for $x + z$, we get

$$n - y > 2y.$$

Rearranging terms gives Condition 1D(a). Moreover, in order that *Z* gains these first preferences rather than *Y,* these $(y - z)$ new first preference ballots for *Z* must all come from the x_z ballots that initially rank *Z* rather than *Y* second,, giving us Condition 1D(b). Condition 1D as a whole is necessary and sufficient for B' to put *Z*, rather than *Y*, in a runoff with *X*.

The Type 2 condition requires that *X* actually beat *Z* in this runoff. By Lemma 1c, *X* must beat *Z* under *B*, i.e., $z + y_z > x + y_x$, but *X* must still beat *Z* after $(y - z)$ first preference ballots shift from *X* to *Z*; that is,

$$x - (y - z) + y_x > z + (y - z) + y_z.$$

Removing parentheses, rearranging, and cancelling gives

$$x + z + y_x - y > y + y_z.$$

Substituting $n - y$ for $x + z$ and $-y_z$ for $y_x - y$, this expression simplifies to $y + y_z < n/2$.

Since $z < y < n/3$, we have the following:

Corollary 2.1. A ballot profile *B* under is vulnerable to Downward Monotonicity Failure only if the IRV winner is not the Plurality Winner.

Since *X* is not the IRV winner under *B*, it cannot be a Majority Winner, i.e., $x < n/2$. The first part of Condition 1D can be restated as $x + z > 2n/3$ or $3(x + z) > 2n$. Given that $x < n/2$, $3(n/2 + z) > 2n$. This simplifies to $z > n/6$, so more generally:

Corollary 2.2. A ballot profile *B* is vulnerable to Downward Monotonicity Failure only if $n/2 > n(\text{PW})$ and $n(\text{PL}) > n/6$.

Thus Downward Monotonicity Failure can occur in less closely contested elections than Upward Failure requires — perhaps in elections in which the weakest candidate gets only about 17% of the vote.

A ballot profile *B* in which *Z* is the Plurality Loser and *X* is not the IRV winner is vulnerable to Downward Monotonicity Failure only if *Y* beats *X* and *X* beats *Z*, so more generally:

Corollary 2.3. A ballot profile *B* is vulnerable to Downward Monotonicity Failure only if (i) the IRV winner is the Condorcet winner or (ii) there is a Condorcet cycle.

5. Double Monotonicity Failure

First, we take note of an obvious relationship between Upward and Downward Monotonicity Failure. Consider a ballot profile *B* that is vulnerable to Upward Monotonicity Failure with respect to companion profile B' as specified by Proposition 1. Then profile B' is clearly vulnerable to Downward Monotonicity Failure with respect to profile *B* as specified by Proposition 2. That is to say, ballot profiles that are vulnerable to Monotonicity Failure come in companion pairs, one vulnerable to Upward and the other to Downward Monotonicity Failure. Thus, Upward and Downward Monotonicity Failure are in some sense the same phenomenon (Note, however, that ballot profiles do not pair off as *unique* companions.)

A further and more subtle question is whether a single ballot profile *B* can be simultaneously vulnerable to both Upward and Downward Monotonicity Failure. Call this vulnerability to *Double Monotonicity Failure*. Such a ballot profile must satisfy both Conditions 1U and 1D and also both Conditions 2U and 2D.

PROPOSITION 3A. There exist ballot profiles that are vulnerable to Double Monotonicity Failure.

Consider the following ballot profile in which *X* beats *Y*, *Y* beats *Z*, and *Z* beats *X*, and the IRV winner is the Plurality Runner-Up *Z*:

38	32	30
X	*Z*	*Y*
Y	*X*	*Z*
Z	*Y*	*X*

The profile is vulnerable to Upward Monotonicity Failure: if 9 of the 38 *XYZ* voters move *Z* to the top of their ballots, *X* becomes the Plurality Loser instead of *Y*, and *Z* then loses to *Y* in the runoff, so *Y* becomes the IRV winner. At the same time the profile is vulnerable to Downward Monotonicity Failure: if 3 of the 38 *XYZ* voters drop *X* to second or third preference, *X* remains the Plurality Winner but *Y* becomes the Plurality Runner-Up, and *X* then beats *Y* in the runoff, so *X* becomes the IRV winner.

The fact that this profile produces a Condorcet cycle is not coincidental.

PROPOSITION 3B. A ballot profile *B* is vulnerable to Double Monotonicity Failure if and only if

(a) there is a Condorcet cycle,
(b) the IRV winner is the Plurality Runner-Up, and
(c) the Plurality Loser beats the Plurality Winner.

The conjunction of Corollaries 1.3 and 2.3 implies that, if ballot profile *B* is vulnerable to both Upward and Downward Monotonicity Failure, there is a Condorcet cycle (specifically the Plurality Loser beats the Plurality Winner, the Plurality Winner beats the Plurality Runner-Up, and the Plurality Runner-Up beats the Plurality Loser). Corollaries 1.1 and 2.1 together imply that the IRV winner must be the Plurality Runner-Up under *B*.

6. Monotonicity Failure with 'Random' Ballot Profiles

We now examine a large and diverse sample of 128,000 randomly generated IRV elections, i.e., ranked ballot profiles. These and other simulations discussed here were conducted at the level of an election, not an individual voter, on the assumption that there were approximately 30 million voters. Over all elections in this set of 'random' simulations, all elections voters were equally likely to cast any one of the six possible IRV ranked ballots. But in any particular election this was not true, so these ballot profiles definitely are not drawn from an 'impartial culture' — an assumption we take up in Section 9. In each ballot profile, the number of first preferences for candidate *X* was drawn from a normal distribution with a mean of 5 million and a standard deviation of 1.2 million, subject to the constraint that $x \geq 0$, and likewise for candidates *Y* and *Z*. Then the number of ballots ranking *X* first and *Y* second was drawn from a normal distribution with a mean of $x/2$ and a standard deviation of $x/6$, subject to the constraint that $0 \leq x_y \leq x$, with $x_z = x - x_y$, with Z ranked second in the remaining ballots. Second preferences on the ballots ranking *Y* and *Z* first were determined in like manner.

[DISCUSSION OF TABLES 1-8]

[DISCUSSION OF FIGURES 1-2]

7. Monotonicity Failure with Single-Peaked Preferences

Single-peakedness means that there is one candidate that no ballot ranks last (and so may also be characterized as *bottom-restricted* preferences). This candidate may be thought of as 'centrist' in his ideological or policy positions relative to the other two candidates, who in turn are (relatively) 'extreme' but in opposite directions (e.g., one to the 'left' and the other to the 'right' of the centrist candidate). Given single-peakedness and labeling the three candidates as Left, Center, and Right, 'admissible' ballot profiles include only four of the six possible rankings of the three candidates, namely LCR, CLR, CRL, and RCL, while LRC and RLC are 'inadmissible' rankings, because they rank candidate C last.

In general, single-peaked ballot profiles can be characterized in terms of three parameters: the proportion of voters who rank the centrist candidate first, the relative balance between the two sets of who rank the left and right candidates first, and the relative balance among centrist voters with respect to their second preferences. However, for the purpose of stating logical propositions, we need to distinguish only between two circumstance: whether either extreme candidate is a Majority Winner or not. Lemma 2 rounds up some elementary propositions concerning single-peaked preferences.

Lemma 3. If voter preferences expressed on a ballot profile are single-peaked:

(a) an extreme candidate is a Condorcet Winner if and only if he is a Majority Winner;
(b) otherwise the centrist candidate is a Condorcet winner;
(c) a Condorcet cycle cannot occur, and
(d) the IRV winner is the Condorcet Winner unless the Condorcet Winner is the Plurality Loser (necessarily the centrist candidate),
(e) in which case the IRV winner is the extreme candidate that beats the other extreme candidate.

Proposition 4A. If voter preferences expressed on ballot profile *B* are single-peaked, *B* is vulnerable to Upward Monotonicity Failure if and only if:

(a) n (PL) > $n/4$, and
(b) the Plurality Loser is the Condorcet Winner.

The first condition is simply Condition 1U, which implies there is no Majority Winner, which in turn implies by Lemma 3 that the centrist candidate Condorcet winner who is stipulated to be the Plurality Loser, which in turn implies Condition

Thus, put more directly, Proposition 4A says that a single-peaked ballot profile is vulnerable to Upward Mononicity Failure if and only if the centrist candidate is the first preference on more than 25% of the ballots but is still the Plurality Loser.

Proposition 4B. If voter preferences expressed on ballot profile *B* are single-peaked, *B* is not vulnerable to Downward (or Double) Monotonicity Failure.

If voter preferences over candidates expressed under ballot profile *B* are single-peaked, the conditions for Downward Monotonicity specified in Proposition 2 cannot be simultaneously fulfilled. Proposition 2 stipulates that *X* is not the IRV winner, *Z* is the Plurality Loser, and (by implication)

Y is the IRV winner. Suppose that *Y* is an extreme candidate. But then *Y* can be the IRV winner only if *Y* is a Majority Winner, i.e., $y > n/2$, contradicting Condition 1D that $y < n/3$. Therefore, *Y* must be the centrist candidate. But this cannot be true either, because single-peakedness then implies that $x_z = 0$, contradicting Condition 1D(ii), i.e., $x_z > y - z$.

Proposition 4B might seem to be contradicted by the fact that a single-peaked profile *B* may by Proposition 4A be vulnerable to Upward Monotonicity Failure with respect to a companion profile B', so B' must be vulnerable to Downward Monotonicity Failure. This is true, but profile B' is not itself single-peaked, so Proposition 4B is not contradicted.[6]

Table 9 shows results comparable to Tables 1-8 with respect to randomly simulated single-peaked ballot profiles, but it displays results for three separate sets of 128,000 profiles. Set (1) was generated in the same manner as the 'random' profiles, except that all ballots with L or R ranked first were assigned C as a second preference. In the other two sets, one candidate was less popular than the two others, with an average of 3 million first preferences while the two others average 6 million. In set (2) the less popular candidate was the centrist candidate and set (3) an extreme candidate.

[DISCUSSION OF TABLE 9, WHICH WILL BE CONVERTED INTO MULTIPLE CROSS-TABULATIONS IN THE MANNER OF TABLES 1-8]

[6] However, as the make-believe sequel to the 2009 Burlington mayor election illustrates, we can find a profile B' that is single-peaked (i.e., with Progressive replacing some Republicans) and that — in the spirit, but not the literal definition, of monotonicity failure — is a companion to *B*.

8. Monotonicity Failure with Clone Candidates

Consider a three-candidate election in which two candidates C_1 and C_2 have similar policy positions or otherwise appeal to same group of voters, while a third candidate E has distinct policy position or otherwise appeals to a different group of voters. Thus there are two distinct sets of voters with substantially opposed preferences: those who prefer both C_1 and C_2 to E and those who prefer E to both C_1 and C_2. However, voters in both groups may have either preference between C_1 and C_2.[7]

This situation can be characterized in several ways. First, C_1 and C_2 may be called (near) *clone* candidates and E may then be called *exceptional* or *extreme* candidate.[8] Second, and parallel to the characterization of single-peaked preferences under which there is one candidate that no one ranks lowest, in this case there is one candidate, namely E, whom no one ranks middle so, as with single-peakedness, ballot profiles include only four of the six possible rankings of the three candidates but, whereas single-peaked preferences are bottom-restricted, these preferences are *middle-restricted*.

The case in which E supporters are a large minority, and specifically when $n/3 < x < n/2$, is of special interest. If the C supporters are sufficiently equally divided between the two clones with respect to their first preferences, candidate E may be the Plurality Winner (and would be elected under Plurality Voting) even though E is also the Condorcet Loser. In this case, the C supporters constitute a majority vulnerable to *vote splitting*, since either can win if the other is not a candidate but, if both are candidates, each spoils the other's chance of election. One appeal of IRV is that it resolves this problem to the advantage of the majority of voters favoring the clone candidates, because at least one clone must get into the instant runoff, where it defeats E and thereby becomes the IRV winner. In effect, the 'first preference' component of IRV functions as an (open) 'primary' for the C voters, determining which clone candidate goes into (and wins) the 'general election' (i.e., the instant runoff) against E. But this advantage of IRV comes at some cost, namely the possibility of (upward) monotonicity failure.

First we round up several well-known or self-evident points in the form of another lemma.

Lemma 4. Given a ballot profile expressing middle-restricted preference,

(a) there is no Condorcet cycle;
(b) the IRV winner is the extreme candidate X if and only if X is the Majority Winner;
(b) if X is not a Majority Winner, it is the Condorcet Loser; in which case
(c) the IRV winner is the Condorcet Winner, namely the clone candidate that beats the other clone candidate.

From Proposition 1, we know that a ballot profile B under which X is the IRV winner and Z is the Plurality Loser is vulnerable to Upward Monotonicity Failure if and only if $z > n/4$ (Condition

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[7] That is, we do not assume that preferences are also single-peaked — whatever considerations lead voters to have conflicting preference between the clones are different in nature from those that lead voters to have conflicting preferences between X and the clones.

[8] Candidates are exact clones when all voters are indifferent between them, but IRV ballots do not allow the expression of indifference.

1U) and Z beats Y (Condition 2U). With middle-restricted preferences, candidate E must be either a Majority Winner or the Condorcet Loser. In the former event, B cannot be vulnerable to Upward Monotonicity Failure by Corollary 1.2, so we can state the following:

Proposition 5A. A middle-restricted ballot profile B is vulnerable to Upward Monotonicity Failure if and only if

(a) the Plurality Loser is a clone candidate C;
(b) $c > n/4$; and
(c) C beats the other clone candidate.

Since (a) and (b) together imply that there is no Majority Winner, Proposition 5A is the same as Proposition 4A, with the added stipulation that the Plurality Loser must be one of the two clone candidates. Note that candidate C must be disproportionately favored as the second preference of E supporters in order for (c) to hold.

We now consider the possibility of Downward Monotonicity Failure in the context of middle-restricted preferences. Proposition 2 stipulates that Z is the Plurality Loser and Y is the IRV winner. Given middle-restricted preferences without a Majority Winner, Lemma 3 says that the IRV winner Y must be one of the two clone candidates by Lemma 3. The next question is whether the other clone candidate is X or Z. Suppose X is the other clone and Z is extreme. Then $x_z = 0$; but, given that Z is the Plurality Loser, $y - z > 0$, so Condition 1D(b) cannot hold. So if B is vulnerable to Downward Monotonicity Failure, Z must be the other clone candidate, making X extreme. This implies that $y_z = y$, so Condition 2D becomes $2y < n/2$ or $y < n/4$. But Z is the Plurality loser, so $z < y < n/4$ and $x > n/2$, making X the Majority Winner and contradicting the stipulation that Y is the IRV winner. Thus, given middle-restricted preferences, the stipulations and the conditions of Proposition 2 cannot all be simultaneously met, giving us the following:

Proposition 5B. If the voter preferences expressed on ballot profile B are middle-restricted, B cannot be vulnerable to Downward (or Double) Monotonicity Failure.

[DISCUSSION OF TABLE 10, WHICH WILL BE COMPLETED AND CONVERTED INTO MULTIPLE CROSSTABULATIONS IN THE MANNER OF TABLES 1-8]

Single-peaked (bottom-restricted) and middle-restricted preferences are examples of *value-restricted* preferences (Sen, 1966). Clearly there is one other category of value-restricted preferences, namely top-restricted (or 'single-caved') preferences, i.e., the existence of a candidate that no one ranks first. But if only two candidates are ranked first, one or the other is a Majority Winner, so we have the following:

Proposition 6. If the voter preferences expressed on ballot profile B are top-restricted, B is not vulnerable to either Upward or Downward Monotonicity Failure.

9. Monotonicity Failure in an Impartial Culture

Many social choice analyses assume that preference profiles are drawn from an *Impartial Culture*, in which voters cast *independent* random ballots — that is, each voter is equally likely to cast a ballot ranking the three candidates in any of the six possible ways. Thus, if the number of voters is reasonably large, $x_y \approx x_z \approx y_x \approx \ldots\ z_y \approx n/6$ and $x \approx y \approx z \approx n/3$.

In a sense, the assumption that preferences are drawn from Impartial Culture is the most 'neutral' one that can be made about voter choice, and as such it provides the basis for many probability calculation in social choice and voting power theory. But we should recognize that this assumption also implies that almost all elections are extraordinarily close. Moreover, in a specific sense the Impartial Culture assumptions the probability of Condorcet cycles (Tsetlin et al., 2003), and Corollaries 1.3 and 2.3, Propositions 3A and 3B, and various simulation results indicate that the existence of Condorcet cycles increases the probability of monotonicity failure.

With this proviso in mind, let us consider the likelihood that a ballot profile is vulnerable to Upward Monotonicity Failure, given an Impartial Culture with many voters.

Since $z \approx n/3$, C1U is (almost) always met, and vulnerability depends (almost) entirely on whether *Z* beats *X*, as stipulated by C2U. In an Impartial Culture, the *unconditional* probability that one candidate beats another must be 0.5 but, given that *Z* is the Plurality Loser, *Z* would be expected to beat *X* (or *Y*) less than half the time. The simulation results presented here below indicate that in an impartial culture the Plurality Winner beats the Plurality Runner-Up about 75% of the time, the Plurality Runner-Up beats the Plurality Loser about 75% of the time, and the Plurality Winner beats the Plurality Loser about 90% of the time. They also show that the Plurality Runner-Up is the IRV winner about 26% of the time. Putting these statistics together, we can anticipate that C2U is met about 12% of the time and therefore that in an Impartial Culture about 12% of ballot profiles are vulnerable to Upward Montonicity Failure.

Condition 1D(a) requires that the Plurality Runner-Up (and IRV winner) *Y* have less than one-third of the first preferences. Overall we would expect this to be true about half the time but, by virtue of being the IRV winner, *Y* is 'stronger' than the 'typical' Plurality Runner-Up. We can therefore expect that *y* is closer to *x* than to *z* and therefore greater than $n/3$ more often than not, but it is hard to anticipate how much more. Condition 1D(b) is (almost) always met, since x_z is almost always on the order of $n/3$, and $y - z$, while by definition always positive, is likely to be small. At first blush, we would expect that $y + y_z \approx n/2$, so that C2D would hold about half the time. But again, *Y* is 'stronger' than the 'typical' Plurality Runner-Up, so C2D presumably holds less than half the time, but again it is hard to anticipate how much less.

I simulated 128,000 three-candidate IRV elections with ballot profiles drawn from an Impartial Culture with 30 million voters.[9] Some of the resulting data is summarized in Table 11. It can be seen from the first row of the table (pertaining to all 128,000 ballot profiles) that the

[9] The simulations took place at the level at the election, not the individual voter. Each of the six ballot rankings was drawn from a normal distribution with a mean of 5 million and a standard deviation equal to the square root of 1.25 million or 1118, i.e., the normal approximation of the binomial distribution with $p = 0.5$.

expectations set out above are generally borne out. Accordingly, about 12.0% ballot profiles are vulnerable to Upward Monotonicity Failure and about 4.8% to Downward Monotonicity Failure. Since about 1.8% are vulnerable in both respects, about 15.0% of the ballot profiles in an Impartial Culture are vulnerable to at least one form of monotonicity failure.

In so far as IRV is deemed superior to Simple Plurality voting (i.e., election of the Pluraility Winner), the 26.1% of the profiles in which IRV produces a different winner (namely the Pluraility Runner-Up) from Simple Plurality, may deserve special scrutiny. The second and third rows of Table 1 compare the frequency of Monotonicity Failure under IRV when IRV elects the Plurality Winner (W1) and when it elects the Plurality Runner-Up (W2). The probability of Upward Monotonicity Failure is almost twice as great in the latter case than overall. Consistent with Corollary 2.1, Condition 2D(a) holds only if the IRV winner is the Plurality Runner-Up

[DISCUSSION OF TABLE 11, WHICH WILL BE CONVERTED INTO MULTIPLE CROSS-TABULATIONS IN THE MANNER OF TABLES 1-8]

10. Monotonicity Failure in English General Elections: 1992-2010

[PRESENTATION AND DISCUSSION OF TABLE 12, WHICH WILL BE CONVERTED INTO MULTIPLE CROSSTABULATIONS IN THE MANNER OF TABLES 1-8]

11. Concluding Remarks

References

1(a) All		C1U		
n = 128,000		No	Yes	Total
C2U	No	43.8%	42.5%	8.6%
	Yes	1.9%	11.8%	13.7%
	Total	45.7%	54.3%	100.0%

(1b) All		C1DA		
n = 128,000		No	Yes	Total
C1DB	No	45.4%	*	45.5%
	Yes	47.9%	6.6%	54.5%
	Total	93.3%	6.7%	100.0%

* 60 cases ≈0.047%

1(c) All		C1D		
n = 128,000		No	Yes	Total
C2D	No	83.3%	2.5%	86.2%
	Yes	9.6%	4.2%	13.8%
	Total	93.4%	6.6%	100.0%

(1d) All		UMF		
n = 128,000		No	Yes	Total
DMF	No	85.8%	10.0%	95.8%
	Yes	2.4%	1.7%	4.2%
	Total	88.2%	11.8%	100.0%

Table 1 Frequency of 'Random' Ballot Profile Meeting Monontonicity Conditions

2(a) IRV ≠ PW		C1U		
n = 27,287		No	Yes	Total
C2U	No	26.0%	45.0%	71.0%
	Yes	4.1%	25.0%	29.0%
	Total	30.1%	69.9%	100.0%

2(b) IRV ≠ PW		C1DA		
n = 27,287		No	Yes	Total
C1DB	No	9.7%	0.2%	9.9%
	Yes	59.0%	31.1%	90.1%
	Total	68.7%	31.3%	100.0%

2(c) IRV ≠ PW		C1D		
n = 27,287		No	Yes	Total
C2D	No	48.9%	11.5%	60.4%
	Yes	20.0%	19.6%	39.6%
	Total	68.9%	31.1%	100.0%

2(d) IRV ≠ PW		UMF		
n = 27,287		No	Yes	Total
DMF	No	63.6%	16.8%	80.4%
	Yes	11.4%	8.2%	19.6%
	Total	75.0%	25.0%	100.0%

Table 2. Frequency of 'Random' Ballot Profile Meeting Monontonicity Conditions when the IRV Winner is Not the Plurality Winner

25-50% Bound		C1U		
n = 69,469		No	Yes	Total
C1U	No	0.0%	78.3%	78.3%
	Yes	0.0%	21.7%	21.7%
	Total	0.0%	100.0%	100.0%

25-50% Bound		C1DA		
n = 69,469		No	Yes	Total
C1DB	No	20.8%	0.1%	20.9%
	Yes	67.5%	11.6%	79.1%
	Total	88.3%	11.7%	100.0%

25-50% Bound		C1D		
n = 69,469		No	Yes	Total
C2D	No	74.1%	4.3%	78.4%
	Yes	14.3%	7.3%	21.6%
	Total	88.4%	11.6%	100.0%

25-50% Bound		UMF		
n = 69,469		No	Yes	Total
DMF	No	74.2%	18.5%	92.7%
	Yes	4.1%	3.2%	7.3%
	Total	78.3%	21.7%	100.0%

Table 3. Frequency of 'Random' Ballot Profile Meeting Monontonicity Conditions when $n/4 < n(PL) < n(PW) < n/2$

Condorcet Cycle		C1U		
n = 12,316		No	Yes	Total
C2U	No	0.0%	0.0%	0.0%
	Yes	16.9%	83.1%	100.0%
	Total	16.9%	83.1%	100.0%

Condorcet Cycle		C1DA		
n = 12,316		No	Yes	Total
C1DB	No	0.0%	0.0%	0.0%
	Yes	77.0%	23.0%	100.0%
	Total	77.0%	23.0%	100.0%

Condorcet Cycle		C1D		
n = 12,316		No	Yes	Total
C2D	No	52.6%	4.2%	56.7%
	Yes	24.4%	18.8%	43.3%
	Total	77.0%	23.0%	100.0%

Condorcet Cycle		UMF		
n = 12,316		No	Yes	Total
DMF	No	16.2%	65.0%	81.2%
	Yes	0.7%	18.1%	18.8%
	Total	16.9%	83.1%	100.0%

Table 4. Frequency of Cyclical ‘Random’ Ballot Profile Meeting Monontonicity Conditions

IRV & Bound		C1U		
n = 19,084		No	Yes	Total
C2U	No	36.6%	15.7%	52.2%
	Yes	21.2%	26.6%	47.8%
	Total	57.8%	42.2%	100.0%

IRV & Bound		C1DA		
n = 19,084		No	Yes	Total
C1DB	No	1.9%	0.2%	2.2%
	Yes	55.6%	42.2%	97.8%
	Total	57.5%	42.5%	100.0%

IRV & Bound		C1D		
n = 19,084		No	Yes	Total
C2D	No	36.6%	15.7%	52.2%
	Yes	21.2%	26.6%	47.8%
	Total	57.8%	42.2%	100.0%

IRV & Bound		UMF		
n = 19,084		No	Yes	Total
DMF	No	49.4%	24.0%	73.4%
	Yes	14.9%	11.7%	26.6%
	Total	64.3%	35.7%	100.0%

Table 5. Frequency of 'Random' Ballot Profile Meeting Monontonicity Conditions when the IRV Winner is Not the Plurality Winner and First Preferences Fall within the 50%-25% Bound

Cycle & Bound		C1U		
n = 10,231		No	Yes	Total
C2U	No	0.0%	0.0%	0.0%
	Yes	0.0%	100.0%	100.0%
	Total	0.0%	100.0%	100.0%

Cycle & Bound		C1DA		
n = 10,231		No	Yes	Total
C1DB	No	0.0%	0.0%	0.0%
	Yes	73.6%	26.4%	100.0%
	Total	73.6%	26.4%	100.0%

Cycle & Bound		C1D		
n = 10,231		No	Yes	Total
C2D	No	46.9%	4.6%	51.5%
	Yes	26.7%	21.8%	48.5%
	Total	73.6%	21.8%	100.0%

Cycle & Bound		UMF		
n = 10,231		No	Yes	Total
DMF	No	0.0%	78.2%	78.2%
	Yes	0.0%	21.8%	21.8%
	Total	0.0%	100.0%	100.0%

Table 6. Frequency of Cyclical 'Random' Ballot Profile Meeting Monontonicity Conditions When First Preferences Fall within the 50%-25% Bound

IRV & Cycle		C1U		
n = 6,181		No	Yes	Total
C2U	No	29.9%	8.3%	38.2%
	Yes	24.3%	37.5%	61.8%
	Total	54.2%	45.8%	100.0%

IRV & Cycle		C1DA		
n = 6,181		No	Yes	Total
C1DB	No	0.0%	0.0%	0.0%
	Yes	54.2%	45.8%	100.0%
	Total	54.2%	45.8%	100.0%

IRV & Cycle		C1D		
n = 6,181		No	Yes	Total
C2D	No	29.9%	8.3%	38.2%
	Yes	24.3%	37.5%	61.8%
	Total	54.2%	45.8%	100.0%

IRV & Cycle		UMF		
n = 6,181		No	Yes	Total
DMF	No	15.1%	47.4%	62.5%
	Yes	1.4%	36.1%	37.5%
	Total	16.6%	83.4%	100.0%

Table 7. Frequency of Cyclical 'Random' Ballot Profile Meeting Monontonicity Conditions when the IRV Winner is Not the Plurality Winner

IRV&Cycle&B		C1U		
n = 5,157		No	Yes	Total
C2U	No	0.0%	0.0%	0.0%
	Yes	0.0%	100.0%	100.0%
	Total	0.0%	100.0%	100.0%

IRV&Cycle&B		C1DA		
n = 5,157		No	Yes	Total
C1DB	No	0.0%	0.0%	0.0%
	Yes	47.5%	52.5%	100.0%
	Total	47.5%	52.5%	100.0%

IRV&Cycle&B		C1D		
n = 5,157		No	Yes	Total
C2D	No	22.6%	9.2%	31.8%
	Yes	24.9%	43.2%	68.2%
	Total	47.5%	52.5%	100.0%

IRV&Cycle&B		UMF		
n = 5,157		No	Yes	Total
DMF	No	0.0%	56.8%	56.8%
	Yes	0.0%	43.2%	43.2%
	Total	0.0%	100.0%	100.0%

Table 8. Frequency of Cyclical 'Random' Ballot Profile Meeting Monontonicity Conditions when the IRV Winner is Not the Plurality Winner and First Preferences Fall within the 50%-25% Bound

Profiles	Cases	W2	Cycle	C1U	C2U	UMF	C1DA	C1DB	C1D	C2D	DMF	2MF	TMF
All (1)	128,000	37.7%	0.0%	54.2%	30.4%	17.9%	16.9%	30.4%	2.2%	18.0%	0.0%	0.0%	17.9%
50-167	111,830	41.2%	0.0%	62.1%	33.0%	20.5%	19.3%	33.0%	2.6%	20.4%	0.0%	0.0%	20.5%
50-25	69,395	42.5%	0.0%	100.0%	33.0%	33.0%	23.8%	33.0%	3.9%	23.7%	0.0%	0.0%	33.0%
W1	79,708	0.0%	0.0%	50.1%	37.4%	20.8%	0.0%	37.4%	0.0%	4.6%	0.0%	0.0%	20.8%
W2	48,292	100.0%	0.0%	61.0%	18.8%	13.1%	44.8%	18.8%	5.9%	40.1%	0.0%	0.0%	13.1%
All (2)	128,000	17.8%	0.0%	15.0%	65.9%	10.7%	6.2%	65.9%	1.1%	4.7%	0.0%	0.0%	10.7%
50-167	73,103	26.6%	0.0%	26.3%	87.4%	18.7%	10.8%	87.4%	2.0%	8.2%	0.0%	0.0%	18.7%
50-25	192,209	41.4%	0.0%	100.0%	71.0%	71.0%	26.7%	71.0%	5.7%	10.1%	0.0%	0.0%	71.0%
W1	105,167	0.0%	0.0%	10.7%	66.5%	10.0%	0.0%	66.5%	0.0%	0.1%	0.0%	0.0%	10.0%
W2	22,833	100.0%	0.0%	34.8%	63.0%	13.6%	34.6%	63.0%	6.3%	25.7%	0.0%	0.0%	13.6%
All (3)	128,000	34.1%	0.0%	15.1%	3.7%	2.2%	9.3%	3.7%	0.3%	12.6%	0.0%	0.0%	2.2%
50-167	73,192	45.7%	0.0%	26.4%	6.3%	3.9%	16.3%	6.3%	0.5%	20.6%	0.0%	0.0%	3.9%
50-25	19,295	40.4%	0.0%	100.0%	14.9%	14.9%	23.6%	14.9%	1.7%	25.6%	0.0%	0.0%	14.9%
W1	84,351	0.0%	0.0%	13.6%	4.8%	2.7%	0.0%	4.8%	0.0%	1.8%	0.0%	0.0%	2.7%
W2	43,649	100.0%	0.0%	17.8%	1.7%	1.3%	27.2%	1.7%	0.8%	33.6%	0.0%	0.0%	1.3%

All(1) On average the three candidates are equally popular
All(2) On average the center candidate is only half as popular as the extreme candidate
All(3) On average one extreme candidate is only half as popular as the other two candidates

Note: The very few tied elections are excluded from these calculations.

Table 9. Monotonocity Failure with Single-Peaked Ballot Profiles

Profiles	Cases	W2	Cycle	C1U	C2U	UMF	C1DA	C1DB	C1D	C2D	DMF	2MF	TMF
All (1)	128,000	49.9%	0.0%	29.4%	16.3%	9.3%	32.2%	50.3%	31.1%	5.2%	0.0%	0.0%	9.3%
50-167	77,047	78.8%	0.0%	48.9%	26.7%	15.5%	53.5%	81.5%	51.7%	8.2%	0.0%	0.0%	15.5%
50-25	37,693	73.9%	0.0%	100.0%	31.17%	31.7%	56.6%	84.1%	54.7%	11.6%	0.0%	0.0%	31.7%
W1	64,180	0.0%	0.0%	15.3%	2.5%	2.1%	0.0%	11.4%	0.0%	7.7%	0.0%	0.0%	2.1%
W2	63,820	100.0%	0.0%	43.7%	30.1%	16.6%	64.5%	89.4%	62.4%	2.6%	0.0%	0.0%	16.6%
All (2)													
Neither													
Both	27,858	100.0%	0.0%	100.0%	38.1%	38.1%	76.6%	94.1%	74.0%	4.7%	0.0%	0.0%	38.1%
W1													
W2													
All (3)													
50-167													
50-25													
W1													
W2													

All(1) All(2) All(3)

Note: The very few tied elections are excluded from these calculations.

Table 10. Monotonocity Failure with Clone Candidates [INCOMPLETE]

Profiles	Cases	W2	Cycle	50-25	C1U	C2U	UMF	C1DA	C1DB	C1D	C2D	DMF	2MF	TMF
All	120,000	26.1%	8.7%	100.0%	100.0%	12.0%	12.0%	7.5%	99.9%	7.5%	12.3%	4.8%	1.8%	15.0%
W1	96,560	0.0%	5.4%	100.0%	100.0%	8.4%	8.4%	0.0%	100.0%	100.0%	4.3%	0.0%	0.0%	8.4%
W2	31,343	100.0%	18.5%	100.0%	100.0%	23.3%	23.3%	30.7%	99.9%	30.7%	37.0%	19.7%	7.4%	36.1%
None	116,926	21.8%	0.0%	100.0%	100.0%	3.7%	3.7%	5.8%	99.9%	5.8%	9.5%	3.3%	0.0%	7.0%
Cycle	11,074	52.5%	100.0%	100.0%	100.0%	99.9%	99.9%	25.7%	99.9%	25.7%	42.0%	21.0%	21.0%	99.9%
Neither	91,307	0.0%	0.0%	100.0%	100.0%	3.1%	3.1%	0.0%	100.0%	0.0%	3.4%	0.0%	0.0%	3.1%
Both	5,811	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	49.0%	100.0%	49.0%	61.8%	40.0%	40.0%	100.0%

Note: The very few tied elections are excluded from these calculations.

Table 11. Monotonocity Failure in an Impartial Culture

Profiles	Cases	W2	Cycle	C1U	C2U	UMF	C1DA	C1DB	C1D	C2D	DMF	2MF	TMF
All	2642	7.8%	0.3%	4.2%	4.5%	1.4%	1.4%	15.2%	1.2%	0.6%	0.3%	0.0%	1.7%
W1	2437	0.0%	0.2%	3.1%	3.8%	1.1%	0.0%	11.6%	0.0%	0.0%	0.0%	0.0%	1.1%
W2	205	100.0%	2.0%	17.6%	13.7%	4.9%	17.6%	58.0%	15.1%	7.8%	4.4%	0.0%	9.3%
None	2634	7.6%	0.0%	4.3%	4.3%	1.4%	1.4%	14.9%	1.2%	0.6%	0.3%	0.0%	1.7%
Cycle	8	50.0%	100.0%	0.0%	100.0%	0.0%	0.0%	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%
50-167	530	22.1%	0.8%	21.1%	19.1%	6.8%	6.8%	51.3%	5.8%	3.0%	1.7%	0.0%	8.5%
50-25	112	32.2%	0.0%	100.0%	32.1%	32.1%	22.3%	75.0%	21.4%	8.0%	7.1%	0.0%	39.3%
Both (1)	117	100.0%	3.4%	30.8%	20.5%	8.5%	30.8%	68.4%	26.5%	13.7%	7.7%	0.0%	16.2%
Both (2)	36	100.0&	0.0%	100.0%	27.8%	27.8%	69.4%	91.7%	66.7%	25.0%	22.2%	0.0%	50.0%

Table 12. Monotonocity Failure in English General Elections: 1992-2010

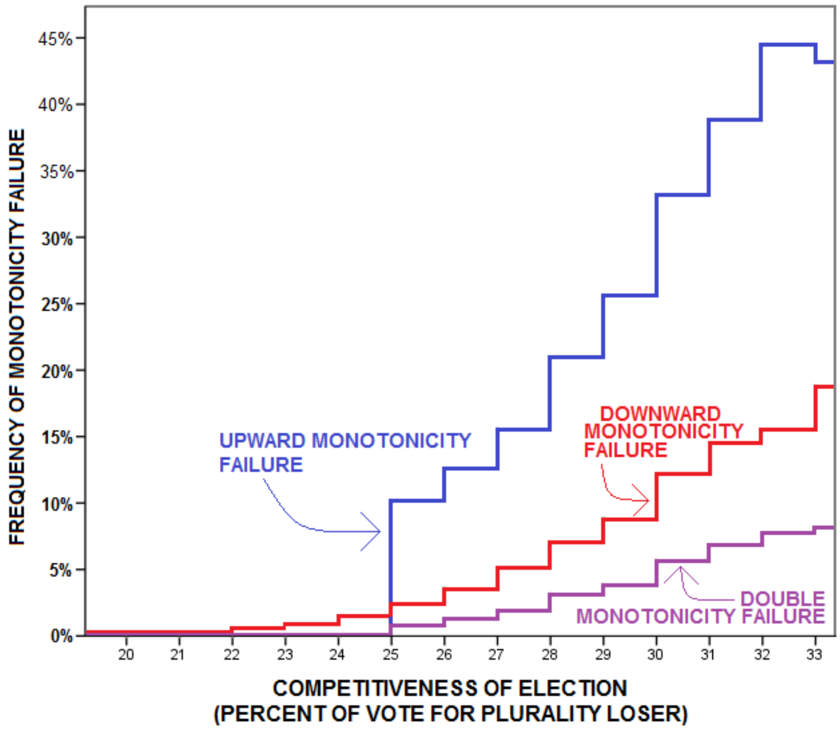

Figure 1. Monotonicity Failure By Election Competitiveness

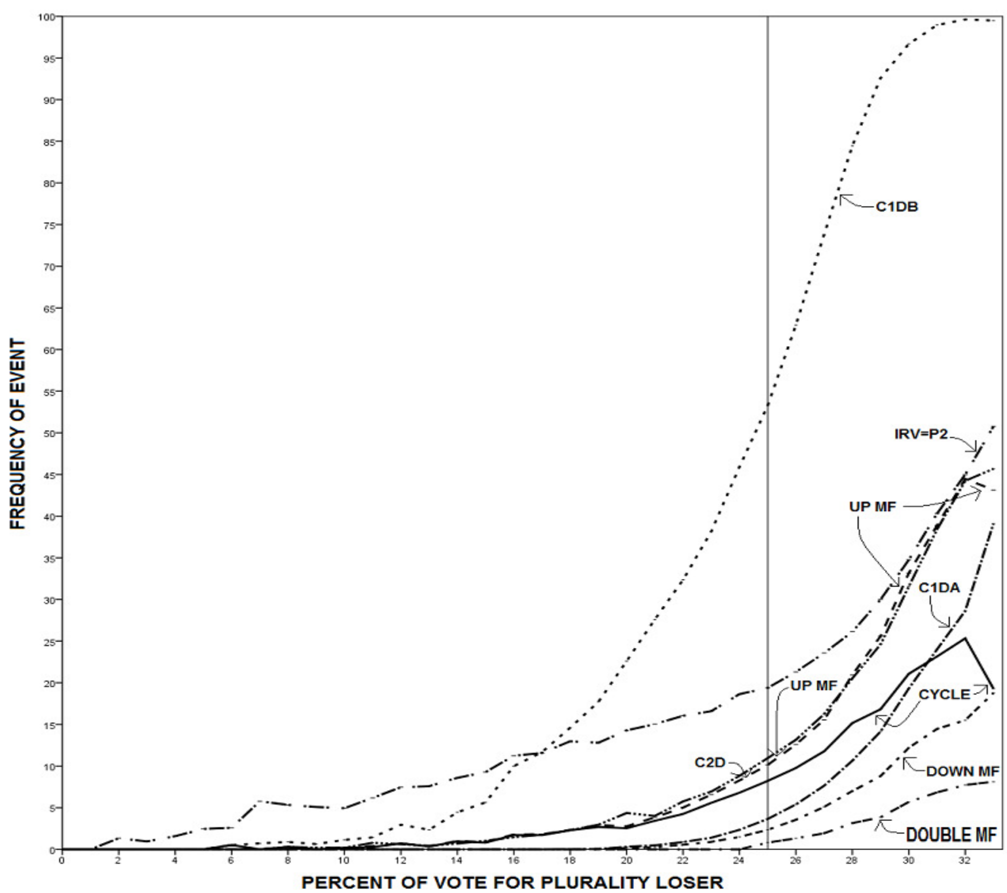

Figure 2.