External Costs and Decision Costs in a Series of Votes

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Abstract

This chapter formalizes external costs and decision costs in a series of votes. The introduction of multiple alternatives affects external costs and decision making costs in two ways. First, multiple alternatives forces us to re-examine the shape of the external cost function and compare it to both the two alternative case (Dougherty and Edward, 2004, p. 171) and the claims made by Buchanan and Tullock (1962). Second, multiple alternatives allows us to formalize decision costs consistent with Buchanan and Tullock's framework. Combined, we are able to make conjectures about the conditions under which specific k-majority rules will minimize total costs. This should improve the classic discussion and help communities establish more appropriate voting thresholds for constitutional and legislative decisions. (JEL D7, C61)

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1 Introduction

How many individuals must agree before a collective decision is imposed on a community? Buchanan and Tullock (1962) raised that question roughly fifty years ago and answered that it depends on how a community weighs decision costs and external costs. At the constitutional stage decision costs are less consequential. Hence, voting rules that produce Pareto superior and Pareto optimal outcomes (or just Pareto optimal outcomes) should be promoted. The only voting rule that could guarantee such results, and minimize external costs, is unanimity rule. At the legislative stage, the optimal k-majority rule may depend on both external costs and decision costs. With decision costs considered, the sum of decision costs and external costs might be minimized somewhere closer to majority rule.

This chapter analyzes the optimal k-majority rule in a context where both external costs and decision costs matter. Although we will discuss these costs in the context of a legislature, the results presented here should apply to constitutional decision making, or any type of decision making, where the optimal k-majority rule depends on both external costs and decision costs.

Previous scholars have considered other types of costs that may be added to the external cost or decision cost functions (Mueller, 1996; Spindler, 1990; Brennan and Hamlin, 2000). The advantage of these studies is that they consider new factors. The disadvantage is that they have not been fully formalized nor have they carefully examined the effects of functional form on their arguments. Dougherty and Edward (2004) have tried to formalize Buchanan and Tullock's argument in the two-alternative case, but they could not adequately analyze decision costs because decision costs do not vary with k when only the status quo and a single proposal are feasible. Furthermore, the Dougherty and Edward do not extend their analysis to a series of votes, which is common in legislatures.

Rather than trying to include new sources of external costs and decision costs, we attempt to assess the claims made by Buchanan and Tullock by creating a framework that can accommodate a series of votes. To conduct our analysis, we formulate a notion of external costs that is analogous to the external costs described by Buchanan and Tullock and by Mueller (2003). We define a BT-loser as an individual who votes for a BT-preferred alternative when society chooses against him/her. We then formalize decision costs in a single round as a constant which makes decision costs in a series largely a function of the probability of passage. This produces several interesting results.

First, if a society values one cost more than the other, then the mere weight it puts on decision costs relative to external costs can make $k = 1$ or $k = N$ optimal. However, even if a society puts no weight on decision costs, it is not always the case that $k = N$ will be uniquely optimal. Because expected external costs largely depend on the probability of passing proposals and the probability of passing proposals is a logistic-type function of k, there is an almost certain probability of passage for k near 0 and an almost certain probability of failure for k near N. Figure 1 depicts this for the case where the probability of favoring the proposal and favoring the status quo are equally likely. Because of the shape of the probability of passage, expected external costs can take a logistic-type shape as well. Without decision costs, a downward step on the right side of the external cost function can make the optimal k-majority rule a range of k-majority rules near $k = N$, rather than a singleton.

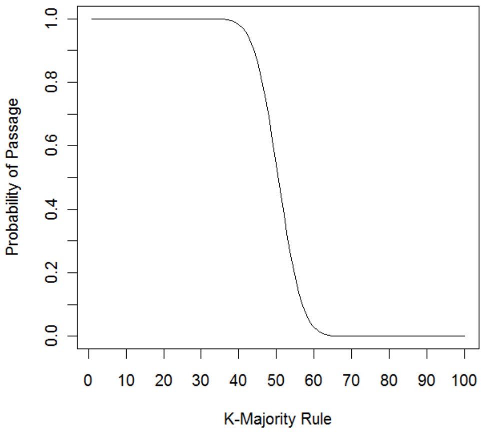

Figure 1: Probability of Passage, $p_1 = p_{-1} = 0.5$, $N = 100$

Second, if external costs and decision costs are equally a concern, then the homogeniety of the society, as depicted by the probability of each preference, can affect the optima. Everything else equal, if a society is extremely homogenous with respect to the decisions it has to make, then more inclusive voting rules might be appropriate. If society is particularly heterogenous with respect to those decisions, then a less-inclusive voting rule may be appropriate.

Third, the ability to create increasingly desirable proposals between rounds can affect the optimal k-majority rule. If political dynamics are such that the probability of passing a proposal quickly increases between rounds, then large k-majority rules may be preferred. If proposals are not increasingly likely to pass in subsequent rounds (or their chances improve only slowly), then moderately smaller k-majorities may be optimal.

Such an analysis can be compared to Mueller's (2003, pp. 76–8) argument that total costs will be minimized at majority rule because of a "kink" in the decision cost function. The difference between our results and those claimed by Buchanan and Tullock, and Mueller, ultimately stems from the fact that for any fixed population, the probability of passage does not decline linearly as k increases. Instead, it decreases in a logistic-type manner, as shown in Figure 1. The difference in shape has important implications for the external cost function, the decision cost function, and many of the more intuitive arguments made by Buchanan and Tullock.

2 Related Literature

Several authors have tried to determine the optimal k-majority rule. At the legislative stage, Buchanan and Tullock argue that the optimal k-majority rule should minimize the sum of external costs and decision costs. Since external costs decrease monotonically as a function of k and decision costs increase monotonically as a function of k, they claim the sum is typically minimized in the neighborhood of majority rule. However, there in nothing unique about majority rule within their analysis. In order for majority rule to be the optimum for a wide class of decisions, they thought that there must exist a kink (i.e., a jump discontinuity) in one of the cost functions at $k/N = N/2$. Mueller (2003, p. 77, n.7) takes up this issue and writes "*If* constitutional conventions choose parliamentary voting rules by weighting the external and decision-making costs of each rule, as Buchanan and Tullock first posited, *then* there is *no way* to explain the ubiquitous use of the simply majority rule *without* the existence of a discontinuity in one of the two curves at $k/N = N/2$" [emphasis in original].

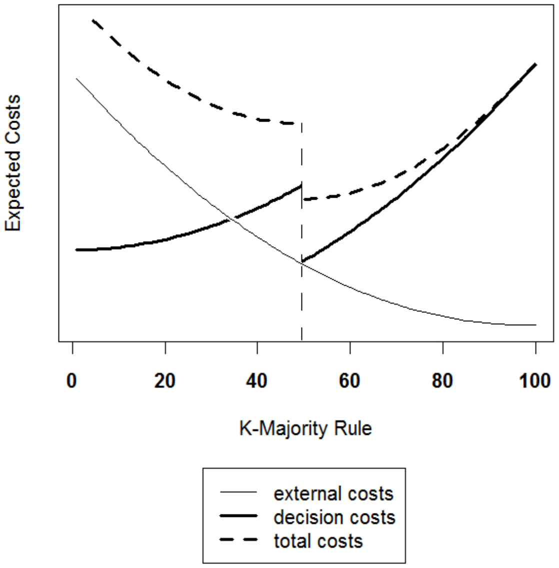

Figure 2: Mueller's Big Jump Discontinuity

Mueller suggests that the possibility of contradictory decisions for $k/N \leq N/2$ can cause a jump discontinuity in the decision cost function at $N/2$. Hence, decision costs decline by a large jump discontinuity just before majority rule, as shown in Figure 2. Combined with a gradually increasing external cost function, such as the one claimed by Buchanan and Tullock, the big jump discontinuity makes majority rule the optimal k-majority rule for a variety of assemblies.

Although a discontinuity such as this suggests that total costs are minimized at majority rule, such a discontinuity may not be necessary for understanding the ubiquitousness of majority rule. This is important because there is no reason to assume that all voting thresholds, less than $N/2$, must produce contradictions. For example, the U.S. Supreme Court requires only four of its nine judges to issues a writ of certiorari. Once the writ is issued, it is not the case that members opposed to the writ can recall it, even if five members are opposed. The rule allows the status quo of no writ to be replaced with a writ, but it does not allow a writ to be recanted by another coalition of equal or larger size once it has been issued. Put differently, there are cases where small k-majority rules avoid self contradictions. In those cases, Mueller's jump discontinuity does not apply.

More importantly, a big jump discontinuity may not be necessary for guaranteeing that total costs are minimized at or near majority rule. As we will later argue, the logistic-type shape of the probability of passage implies a sharp drop in the external cost function. Although the exact location of this drop depends on the preference probabilities, many institutional framers may believe that favoring and opposing a proposal is equally likely. In these cases, the external cost function will typically drop in the neighborhood of majority rule. A variety of upward-sloping decision cost functions would then bring constitutional framers to the conclusion that total costs are minimized at or near majority rule.

3 Definitions

Some simple notation is needed for our analysis. Let N be the number of individuals and M be the smallest majority of those individuals; so that $M = (N+1)/2$ for N odd and $M = (N+2)/2$ for N even. Each individual will have preferences over a set of alternatives $\{w, x, y, z, q\}$. We will reserve the term q for the alternative that is also the status quo (i.e., the existing policy, candidate, or state of affairs). For every pair of alternatives $\{x, y\}$, each individual has one of three preferences: $x \succ_i y$ if and only if he/she strictly prefers x to y; $y \succ_i x$ if and only if he/she strictly prefers y to x; and $x \sim_i y$ if and only if he/she is indifferent between the two alternatives.

We also assume that individuals with a strict preference vote sincerely (i.e., for their most preferred alternative in each pair), and indifferent individuals either "vote abstain" or "not vote," without loss of generality.

Finally, we assume that preferences are randomly drawn from the domain of all possible preferences and that each of the pairwise rankings between any x and y ($x \succ_i q$, $x \sim_i q$, and $q \succ_i x$) occur in the domain with probabilities p_1, p_0, and p_{-1}, respectively.[1] These probabilities reflect uncertainty

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[1] For simplicity, we assume that each individual has the same preference probabilities and that these probabilities are independent. One way to meet these two assumptions is to randomly draw an assembly from a meta-population. For example, a Board of Education randomly drawn from a conservative county may prefer school vouchers to the status quo of publicly funded education with probabilities $p_1 = .60$, $p_0 = .10$, and $p_{-1} = .30$, while a Board of Education

about preferences, which institutional framers would have if they select voting rules before actors or alternatives are fully known.

In a homogenous population, proposals will tend to be either uniformly liked or uniformly disliked. If a proposer tends to propose something in line with their individual or group interests, then p_1 will be large and p_{-1} will be small. In a heterogenous population, a proposer might find it difficult to formulate successful proposals. In other words, p_1 will be small and p_{-1} will be large.

4 One Vote, Two Alternatives

Buchanan and Tullock (1962, p. 45) described external costs as "the costs that an individual expects to endure as a result of the actions of others over which he has no control." To formalize this idea in a two alternative setting with a proposal and a status quo Dougherty and Edward (2004, p. 171) define the concept of a BT-loser. A BT-loser is an individual who votes against society when society chooses a BT-inferior alternative. That is, an individual who votes for proposal x when x is BT-preferred to the status quo q and society chooses q; or an individual who votes for q when q is BT-preferred to x and society chooses x.[2] A BT loser either suffers a loss because society fails to pass a Pareto preferred proposal that he/she wants or because society passes a proposal that makes some individuals, BT losers, worse off.[3] We can think of the former as a case where a government, or other institution, prevents the adoption of a proposal to which no one would object. That creates a certain amount of external cost due to inaction. We can think of the latter as a case where the actions of a government, or other collective, has adopted a proposal that makes at least some individuals worse than they would be if the action was never taken. Dougherty and Edward (2004) define two types of external costs based on BT losers. "Expected BT-loss" is the expected number of BT losers per person

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drawn from a liberal county may prefer the two alternatives with probabilities $p_1 = .30$, $p_0 = .10$, and $p_{-1} = .60$. Stochastic preferences might also assure independence.

[2] Proposal x is BT preferred to status quo q if and only if it is Pareto preferred to q; otherwise q is BT preferred to x.

[3] In describing external costs, Buchanan and Tullock clearly have in mind the number of individuals who will suffer a cost imposed by others. They do not focus on the intensity of preferences nor try to aggregate intensities across individuals (see their pages 64 and 77–80).

whether or not the proposal passes and "expected BT-loss from a passed measure" is the expected number of BT-losers per person from a passed measure. Buchanan and Tullock seem to presuppose that a proposal passes eventually. In much of our analysis, we do not make this assumption.

Dougherty and Edward (2004) derive the following formulas for these two types of external costs. Expected BT-loss under *absolute* k-majority rule:

$$ec_a = \frac{1}{N}\sum_{s=k}^{N-1}\sum_{t=1}^{N-s}\binom{N}{s}\binom{N-s}{t}(p_1)^s(p_{-1})^t(p_0)^{N-s-t}\cdot t + \sum_{s=1}^{k-1}\binom{N}{s}(p_1)^s(p_0)^{N-s}\cdot s. \tag{1}$$

The factor $1/N$ in front of the sum ensures that we are measuring costs per person.

Expected BT-loss under *simple* k-majority rule:

$$ec_s = \frac{1}{N}\sum_{i=0}^{N-1}\sum_{s=\lfloor k-(ik/N)\rfloor+1}^{N-i-1}\binom{N}{i}\binom{N-i}{s}(p_0)^i(p_1)^s(p_{-1})^{N-i-s}\cdot(N-i-s).$$

Let $pass_a$ be the probability that the proposal passes under absolute k-majority rule in the two-alternative case. Expected BT-loss from a passed measure under *absolute* k-majority rule is then:

$$(ec_a|\text{pass}_a) = \frac{1}{N}\frac{\sum_{s=k}^{N-1}\sum_{t=1}^{N-s}\binom{N}{s}\binom{N-s}{t}(p_1)^s(p_{-1})^t(p_0)^{N-s-t}\cdot t}{\sum_{s=k}^{N}\binom{N}{s}(p_1)^s(p_0+p_{-1})^{N-s}}$$

It can also be shown that the probability the proposal passes under *simple* k-majority rule is

$$\text{pass}_s = \sum_{i=0}^{N-1}\sum_{s=\lceil k-(ik/N)\rceil}^{N-i}\binom{N}{i}\binom{N-i}{s}(p_0)^i(p_1)^s(p_{-1})^{N-i-s}.$$

And the Expected BT-loss from a passed measure under *simple* k-majority rule is[4]

$$(ec_s|\text{pass}_s) = ec_s/\text{pass}_s \tag{2}$$

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[4] Dougherty and Edward (2004) provide a different, but equivalent, formalization.

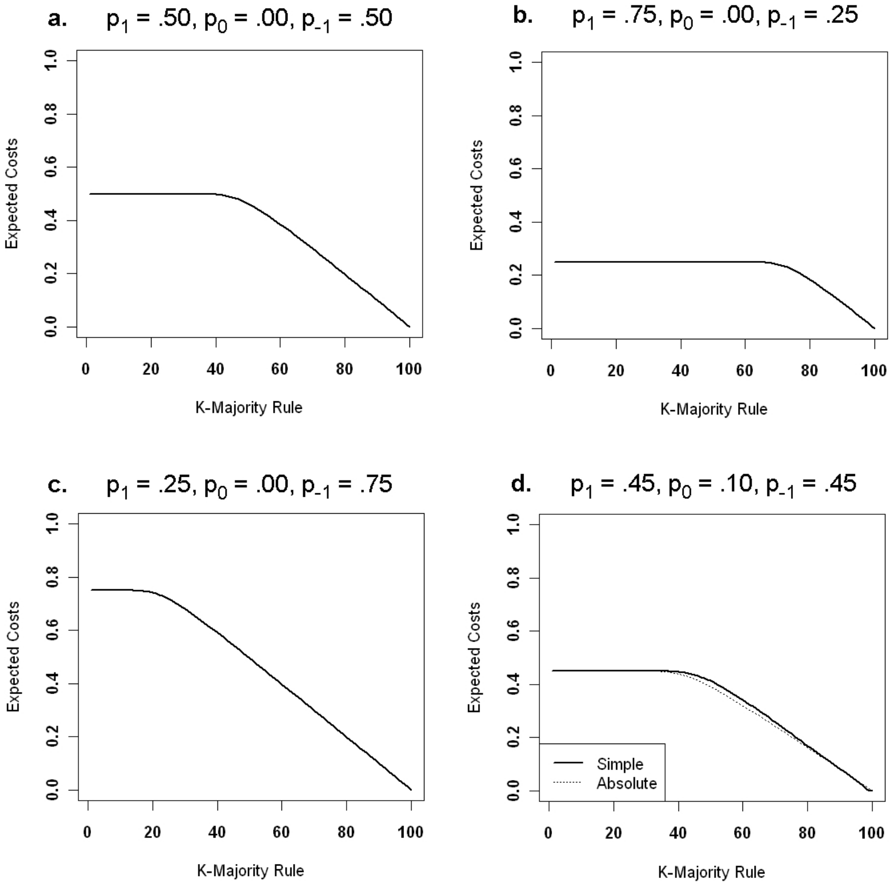

Figure 3: Expected BT-Loss from a Passed Measure, $N = 100$

Unless $p_0 = 0$ or $p_1 = 0$, $(ec_a|\text{pass}_a) \neq ec_a/\text{pass}_a$, because of the presence of the second term in equation (1). The second term captures BT loss from failure to pass a Pareto preferred proposal. An equivalent term does not exist for simple k-majority rules because simple k-majority rules always pass Pareto preferred proposals.

Figure 3 graphs the expected BT-loss from a passed measure for various preference probabilities and $N = 100$. Figure 3a presents a case with no indifferent voters and individuals equally likely to favor and disfavor the proposal. As the frame indicates, external costs behave as Buchanan and Tullock described for $k/N > p_1$ — that is, external costs slowly decline as k increases. For k less than p_1 external costs are largely flat, contrary to their descriptions. The expected BT-loss from a passed measure has a similar shape for other preference probabilities. However, as Figures 3b and 3c indicate, the location of the noticeable decline depends upon the values of p_1 and p_{-1}. If $p_1 > 1/2$, the decline begins at larger values of k. If $p_1 < 1/2$, the decline begins at $k < 50$. This affects the extent to which expected BT-loss from a passed measure behaves as Buchanan and Tullock might suggest. In Figure 3c, the slow decline starts early and most of the figure reflects a gradual decrease as they might claim. However, in Figure 3b, the slow decline starts late and external costs are essentially constant for most k, contrary to their claims. Furthermore, the maximum value of the function equals p_{-1}. The reason is that for low k, where the proposal is almost certain to be pass, the expected number of voters saying nay will be $p_{-1}N$, and thus expected external costs should be $p_{-1}N/N$.

Recall that without indifferent voters (or voters who abstain) simple k-majority rule and absolute k-majority rule behave equivalently. Hence, frames a–c depict the expected BT-loss from a passed measure for both simple and absolute k-majority rules. Allowing the possibility of indifferent voters does not change the general shape of the function. It merely separates the simple and absolute cases. In general simple k-majority rule almost always declines to the right of absolute k-majority rule as faintly shown in Figure 3d for for $40 < k < 70$.

Dougherty and Edward (2004) show that these shapes change fairly substantially if it is not presupposed that the proposal passes. Figure 4 graphs the expected BT-loss for the same preference probabilities as shown in Figure 3. The only difference is that we do not presuppose that the proposal passes. This gives the external cost function more of a logistic-type shape. For large populations, such as $N = 100$, almost all of the decrease in expected BT-loss takes place in a narrow interval centered around $k/N = p_1$. If p_1 is small (as in Figure 4c), then the drop starts at smaller values of k. If p_1 is large (as in Figure 4b), then the drop starts at large values of k. As in the previous figure, the height of the expected loss function is fully determined by p_{-1}.

Keep in mind that the probability of failing to pass a Pareto preferred proposal is a rare occurrence and it cannot occur if $p_0 = 0$. Hence, the marked difference in shapes between Figure 4 and Figure 3

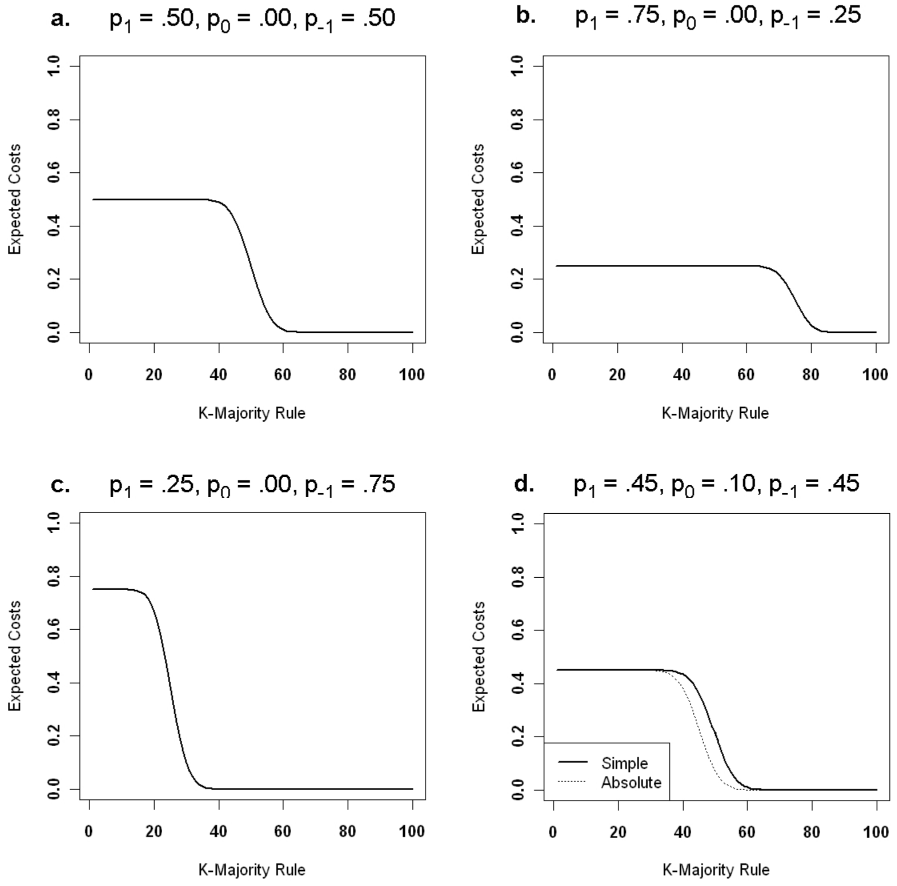

Figure 4: Expected BT-Loss, $N = 100$

is not typically due to the new type of BT loser. Instead it results from the conditional expectation of presupposing that the measure passes. For simple k-majority rules this is operationalized by the inclusion of the probability of passage in the denominator of equation (2). With the denominator, we get shapes like Figure 3a. Without the denominator, we get shapes like Figure 4a. Dividing by the probability of passage is fully responsible for the more gradual decline in the earlier figure.

With indifferent voters absolute and simple k-majority rules do not select equivalently. Instead, simple k-majority rule dips to the right of absolute k-majority rule, as depicted in Figure 4d. This is because for any k, simple k-majority rule is more likely to pass a proposal than absolute k-majority rule, with noticeable differences for values of k/N near p_1.

At this point it would be interesting to include some formalization of decision costs. Decision costs are the "time and effort …required to secure agreement" (Buchanan and Tullock, 1962, p. 68). Buchanan and Tullock argue that decision costs increase with k because larger k-majority rules make it more difficult to formulate *successful* proposals. This is not the case in a two-alternative framework, with a proposal and a status quo. A single proposal prohibits any restructuring of a measure to make it more palatable. Hence, the time and effort required to consider a single proposal would be the same for all k. Decision costs would be constant. In order to allow decision costs to vary with k, we have to consider multiple alternatives in a series.

5 A Series of Votes, Multiple Alternatives

We now extend the analysis to include multiple alternatives in a series of votes. For this part of the analysis assume that an assembly meets to pass a resolution on some issue and that voters are uncertain about the preferences of other voters. A first proposal is made, and if the proposal were put to vote, individuals would vote for or against the proposal with probabilities $p_{1,1}$ and $p_{-1,1}$ respectively. Here $p_{-1,1}$ indicates the probability of opposing the proposal in round 1. For $r > 1$, $p_{1,r}$ (resp. $p_{-1,r}$) is the probability that an individual favors (resp. opposes) the proposal in round r given that the proposals in the previous rounds have been defeated. For simplicity, we assume that there are no indifferent voters. In other words, $p_{0,r} = 0$. Hence, the distinction between simple and absolute is no longer important and the only BT loss comes from a passed proposal. We also assume:

(i) in each round, the probabilities $p_{1,r}$, $p_{-1,r}$ are the same for all individuals; and

(ii) in each round, a voter's preference is independent of the preferences of the other voters.

After the motions is proposed, it is discussed then put forward for a vote. If the motion passes, the process ends. If the motion is defeated, then a new proposal is made with new values for $p_{1,r+1}$ and $p_{-1,r+1}$. To model the notion of continuously improving proposals, we assume that the information gathered in the discussions of the previous proposal and the subsequent vote are enough to make the the new motion more likely to pass than the motion in the previous round (i.e., $p_{1,r} > p_{1,r-1}$). The

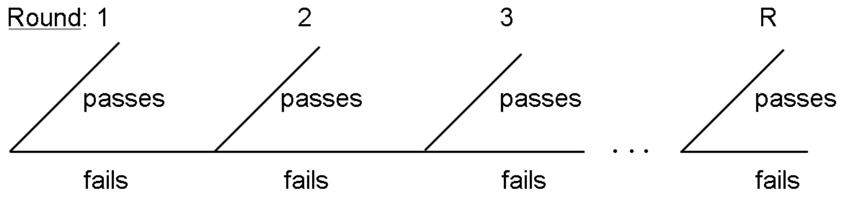

Figure 5: A Finite Series of Votes

procedure is then repeated. In each round a proposal is discussed, followed by either passage of the proposal or a new proposal. In a finite series, successive proposals are made until a proposal passes or the final round R is reached.

The process follows a "successive" voting procedure (Rasch, 2000). That is, the initial status quo q_1 is paired against a proposal x_1 in round 1. If x_1 passes, voting ends. If x_1 fails, q_1 is paired against x_2 in round 2, and so on, for a total of R rounds (see Figure 5). In a successive procedure, voting continues until a proposal passes or the final round R is reached where the proposal can either pass or fail. The successive procedure is widely used by national legislatures in Europe, including the national legislature in France, Germany, Spain, Greece, and Norway, to name a few (Rasch, 2000). A different procedure, often referred to as an amendment procedure (or elimination procedure) is used in the United States, Canada, and Great Britain. The successive procedure need not be limited to legislative decision making. It may also apply to constitutional decision making in small communities and to other types of decisions.

5.1 Decision Costs

To formalize decision costs in this setting, we assume that each round of proposal/discussion imposes the same decision making costs on the assembly, $c > 0$.

We also assume that p_1 increases in each round by an increment, that increment is either α or α/r, where $\alpha > 0$ is some constant. Since $p_1 + p_{-1} = 1$, this means p_{-1} will decrease by the same increment. Assuming α creates a constant increase in the probability of favoring a proposal between rounds. Assuming α/r implies that as r increases the probability of finding a favorable proposal increases at a decreasing rate.[5]

5.2 External Costs

To formalize our notion of external costs, let $E[C_r]$ denote the expected BT loss from round r, and F_r denote the event "the proposal is defeated in round r." By the properties of conditional expectation, we have for $r > 1$

$$\begin{aligned} E[C_r] &= E[C_r|F_1 \cap \ldots \cap F_{r-1}]P[F_1 \cap \ldots \cap F_{r-1}] \\ &= E[C_r|F_1 \cap \ldots \cap F_{r-1}]P[F_1]P[F_2|F_1]\ldots P[F_{r-1}|F_1 \cap \ldots \cap F_{r-2}]. \end{aligned}$$

Under the assumptions described in Section 5,

$$P[F_1] = \sum_{s=0}^{k-1} \binom{N}{s} (p_{1,1})^s (p_{-1,1})^{N-s},$$

and for $1 < j \leq R$,

$$P[F_j|F_1 \cap \ldots F_{j-1}] = \sum_{s=0}^{k-1} \binom{N}{s} (p_{1,j})^s (p_{-1,j})^{N-s};$$

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[5] We have chosen to formalize decision costs as constant across rounds and let the differences between various k depend on the probability of passage. There are certain advantages to such a formalization. First, it is simple. Second, it captures Buchanan and Tullock's notion that decision costs are smaller in more homogenous populations (Buchanan and Tullock, 1962, pp. 115–6), where $p_{1,1}$ should be greater, than in more heterogeneous populations. Third, the formalization is consistent with Buchanan and Tullock's claim that decision costs among k members of a group size N will generally be smaller than unanimity among a group of size k (Buchanan and Tullock, 1962, pp. 106–8). For example, in a series of votes decision costs should be smaller for a voting threshold of 51 members out of a group of 100 than for a voting threshold of 51 members out of 51. This is because there are more combinations of a coalition of 51 members out of 100 than there are combinations of 51 members out of 51 (Buchanan and Tullock, 1962, pp. 106–8). Fourth, because of the associative property of addition and multiplication, c also provides a relative weight between decision costs and external costs in the total cost function. However, there are disadvantages to such a formalization. First, the exact value of c might not be easy to determine. Everything else equal, larger values of c may cause total costs to be minimized at smaller values of k. Second, our formalization of decision costs does not model bargaining and other game-theoretic processes explicitly. It only "assumes" that there is a process that increases the probability of passage each round.

and

$$E[C_r|F_1 \cap \ldots \cap F_{r-1}] \quad = \quad \sum_{s=k}^{N}(N-s) \cdot \binom{N}{s}(p_{1,r})^s(p_{-1,r})^{N-s}.$$

Finally, we define external costs in a series as

$$EC = \sum_{r=1}^{R} E[C_r].$$

In the last equation R could be infinite, which implies that a proposal is never accepted.

6 Results

Figure 6 presents decision costs (thick lines), external costs (thin lines), and total costs (dashed lines) for the initial preference probabilities $p_{1,1} = p_{-1,1} = 0.5$ and $c = .01$. Each frame varies by the increment used to increase $p_{1,1}$ and decrease $p_{-1,1}$ between rounds. In frames a and b, $p_{1,1}$ is increased by a constant. We continue the series until $p_{1,r} = 1.0$. For $\alpha = 0.1$, this implies that the series will last at most six rounds. For $\alpha = 0.001$, the series lasts at most 501 rounds. Hence, a proposal will pass at the end of both sequences.

If a proposal will always pass, why aren't the external costs the same for every k-majority rule? The answer is that smaller k-majority rules will be more likely to pass the proposal early, when $p_{1,r}$ is smaller, $p_{-1,r}$ is larger, and the external external costs associated with $p_{-1,r}$ are high. Larger k-majority rules are more cautious. They are unlikely to pass a proposal until $p_{1,r} > k/N$. In those cases, $p_{1,r}$ is larger, $p_{-1,r}$ is smaller, and external costs associated with $p_{-1,r}$ are small as well. The differences are not in whether the proposal passes by the end of the series. In these two series they certainly will. The differences are whether a k-majority rule will allow a proposal to pass in an early round or will cause the assembly to wait until later rounds when proposals will appeal to a large portion of the population.

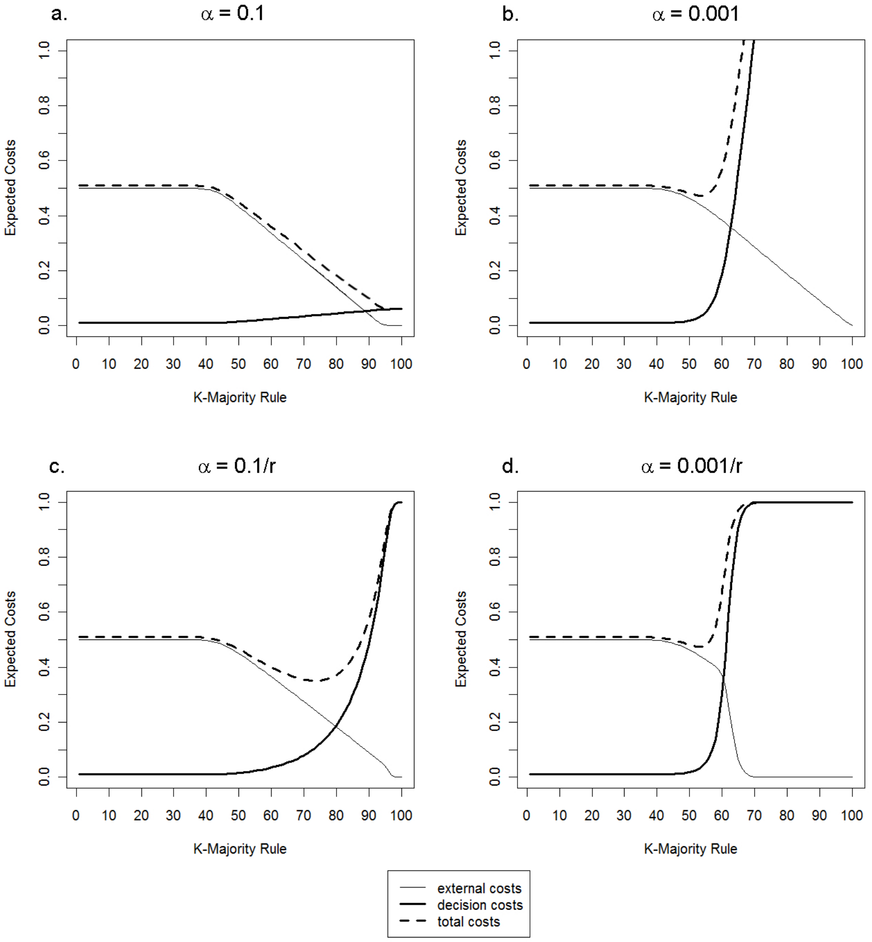

Figure 6: Decision Costs, External Costs, and Total Costs in a Series of Votes, $c = .01$

Note: $p_{1,1} = p_{-1,1} = 0.5$.

Consider frame *a*. In this case, the probability of favoring the proposal increases quite rapidly between rounds. External costs in this frame look surprisingly similar to the expected BT loss from a passed measure for the two-alternative case (Figure 3a). However, unlike the one-shot case, all $k > 92$ produce roughly zero external costs. In addition, decision costs start near .01, for all $k < 45$, because proposals are easily passed for such k. After that point decision costs increase linearly for $45 \leq k \leq 100$. The increase is slow because $p_{1,r}$ increases rapidly to 1 as r increases. This rapid increase means that for larger k, $p_{1,r}$ quickly exceeds k/N, which prevents larger k-majority rules from incurring much decision costs. Because external costs are much larger than decision costs for almost every k, the sum of the two functions is minimized at $k = 99$ and 100. Such a result supports the notion that unanimity rule or a near unanimity rule are ideal (Wicksell [1896] 1967).

Frame b differs from frame a by the rate at which the probability of favoring the proposal increases between rounds and the total number of rounds before $p_{1,r}$ converges to 1. In this case, external costs look fairly similar to those frame a. The major difference between the two frames is the decision costs. Decision costs start at the same location as in frame a, but make a sharp increase shortly after .5. This steep incline is due to the fact that $p_{1,r}$ increases very slowly to 1. The slow increase implies that for $k > 50$ there will be many rounds where $p_{1,r} < k/N$. These are cases where the proposal is very unlikely to pass. Combined with external costs such decision costs creates a unique minimum in the total cost function at $k = 52$. This results supports the notion that majority rule or a near majority rule should be chosen.

Frames c and d depict results for increasing $p_{1,r}$ at a marginally decreasing rate. In these cases, $p_{1,r}$ reaches 1.0 only for extremely large R. Hence, we stop the series after $R = 100$ rounds.[6] The meaningful difference between the result displayed in frames c and d is the rate at which the probability of favoring proposals increases, particularly in earlier rounds.

Frame c shows the results for a slowly increasing α. Again, external costs slowly decline after $k = 45$ similar to the previous frames. Even though $p_{1,100} < 1$, the probability of passing a proposal in 100 rounds or less is roughly 1 for almost all values of k (see Figure 7a) — hence the almost linear decline in external costs. At the same time, decision costs increase more abruptly than in frame a but less abruptly than in frame b, because $p_{1,r}$ increases slower in frame c than in frame a and more quickly than in frame b. Combined, the total cost function is minimized at $k = 74$. In this case, a supermajority rule, such as 3/4ths rule might be recommended for the community.

Finally, frame d displays a case that is identical to the one depicted in frame c except the rate that $p_{1,r}$ increases much slower. After $r = 100$, the cumulative probability of passing a proposal is much smaller for larger values of k as shown in Figure 7b. The difference in the cumulative probability

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[6] For $\alpha = 0.1/r$, $p_{1,r}$ converges to 1 at $r = 226$. For $\alpha = 0.001/r$, $p_{1,r}$ converges to 1 at $r > 100,000$.

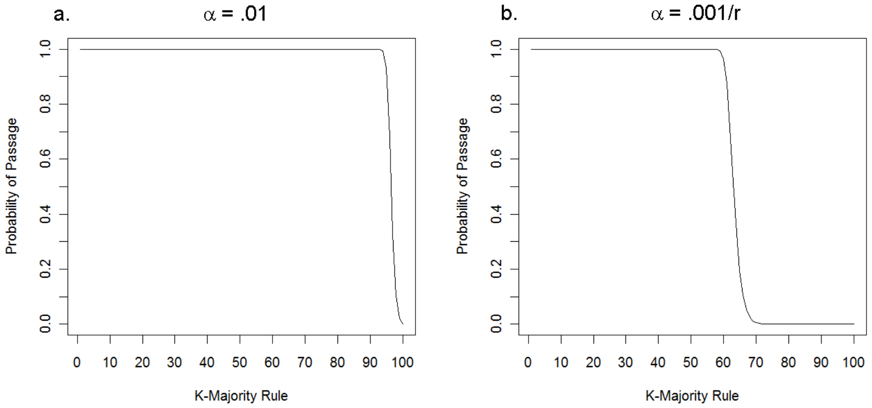

Figure 7: The Probability of Passing a Proposal in 100 Rounds or Less

Note: $p_{1,1} = p_{-1,1} = 0.5$.

of passage produces two noticeable effects. First, the external cost function looks much more like a logistic-type function, as in the two alternative case without a presupposition that the proposal passes (Figure 4a). The reason for this is that a proposal is not likely to pass for $k > p_{1,1}N$. In Figure 7a, a proposal is likely to pass under such k and external costs decline more gradually as a result. Second, the decision cost function in frame d increases sharply, as in frame b or c, but it takes on a more logistic-type shape. Despite these differences the total cost functions in frames b and d are fairly similar. In this case, the optimum is $k = 53$, similar location to frame b.

Of course, different values for decision costs per round, c, will effect total costs. Figure 8 presents results for $c = .15$. Otherwise, the parameters are identical to those depicted in Figure 6a. Note that in this case, total costs are more or less constant for all $k < 52$, making a wide range of less-inclusive k-majority rules optimal. There are two reasons. First, for large k, decision costs dominate external costs in this case, and second, for $k > 45$, decision costs have a moderately steep slope compared to Figure 6a. In general, larger c will imply that the optimum is a less-inclusive voting rules as one might otherwise expect.

Altering the initial preference probabilities, $p_{1,1}$ and $p_{-1,1}$, also has an effect similar to the two-alternative case. If $p_{1,1} = 0.7$ and $p_{-1,1} = 0.3$ (not shown), external costs have an initial value of .3

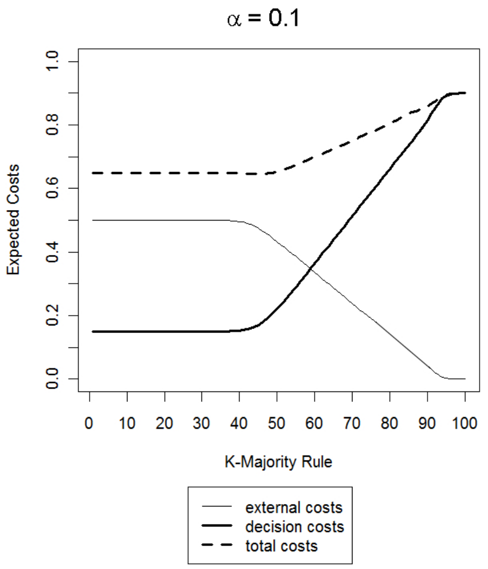

Figure 8: Decision Costs, External Costs, and Total Costs in a Series of Votes, $c = .15$

Note: $p_1 = p_{-1} = 0.5$.

and decline slowly for values of k such that $k \geq 70$ (similar to Figure 3b). If decision costs increase abruptly, as they do for $\alpha = .001$ and $\alpha = .001/r$, then the steep increase will begin at values of k near $k = p_{1,1}(N) = 70$. In these cases, supermajority rules near 75 might be optimal. If decision costs increase slowly as a function of k, as they do for $\alpha = .1$, then the optimal k-majority rule will be a small range near unanimity.

In contrast, if $p_{1,1} = 0.3$ and $p_{-1,1} = 0.7$, external costs will start at .7 and decline slowly for values of k such that $k \geq 30$ (similar to Figure 3c). For sharply increasing decision costs, such as those from $\alpha = .001$ and $\alpha = .001/r$, the optimal k-majority rule would be in the vicinity of $k = 35$. Hence, one implication is that if a society is extremely homogenous from the beginning, it might want to consider a more inclusive k-majority rule. If a society is particularly heterogenous, it might consider a less inclusive k-majority rule.

7 Conclusion

Any formalization of decision costs and external costs should include the likelihood of passing proposals. After all, the probability of passing a proposal affects the time and effort needed to reach an agreement and the ability to impose costs on other individuals. In our model, the probability of passage is is not linear across k. Instead, it is a logistic-type function as depicted in Figure 1. For any given $(p_{1,r},\, p_{-1,r})$ there are k where the proposal will almost certainly pass and and other k where it will almost never pass. As a consequence, there are often horizontal regions in decision costs and external costs functions near the extremes. These horizontal regions provide some of our most interesting, but perhaps most unintuitive, results. For example, unanimity rule may guarantee zero external costs, but that does not mean it should be treated as uniquely minimizing external costs. For all practical purposes, other k-majority rules near unanimity may be equally adept at minimizing external costs in some circumstances. Hence, if we cared solely about external costs we might get a range of optimal voting rules. In this sense, unanimity rule might be an ideal voting rule for the constitutional phase of decision making. However, even without decision costs, it would not be a unique ideal.

In the same vein, $k = 1$ may not be uniquely qualified for minimizing decision costs. A range of k-majority rules may produce roughly the same decision costs as $k = 1$.

In a series of votes, the optimal k-majority rule depends on several factors, one of which is the probability of passage, just described. Another is the ability to create increasingly desirable proposals between rounds. Suppose an institutional framer assumes that favoring and opposing proposals is equally likely. Under such conditions, larger k-majority rules, such as unanimity rule, should be considered if individuals can reformulate proposals in a way that substantially increases the probability of favoring measures between rounds. If the probability of favoring proposals increases quickly, larger k-majority rules will inhibit proposals that hurt minorities without amassing large decision costs. However, if proposals can only be reformulated in a way that slowly increases the probability of favoring a proposal between rounds, then institutional framers might have reason to favor k-majorities near $p_{1,1}(N)$. The slow increase in the probability of favoring a proposal would imply significantly larger decision costs for $k > (.5)N$. If favorable proposals evolve moderately, between the two extremes, then institutional framers may want to adopt supermajority rules, such as 3/4ths rule.

Another factor that affects the optimal k-majority rule is the decision costs per round, c. This variable serves at least two purposes in our formulation: it helps to put decision costs and external costs on a common scale and it provides a relative weighting between the two terms in the total cost function. As should be obvious, for each set of parameters, there will always be a sufficiently large value of c that will make $k = 1$ optimal (perhaps among other k-majority rules in the neighborhood of $k = 1$) and a sufficiently small value of c that will make $k = N$ optimal (perhaps among other k-majority rules in the neighborhood of $k = N$).

Furthermore, the optimal k-majority rule seems to depend on the homogeneity of the society, as depicted by the initial preference probabilities ($p_{1,1}$, $p_{-1,1}$). If a society is extremely homogenous from the beginning so that, say, $p_{1,1} = .7$, then it might want to consider a more inclusive k-majority rule. If a society is particularly heterogenous, especially with respect to the decisions that have to be made, then it might want to consider a less inclusive k-majority rule (similar to our results for $p_{1,1} = .3$). External costs will be greater in the latter case, but these costs will decline quickly and they will be offset by increasing decision costs that can mount up as individuals disagree.

Contrast this result with a claim made by Buchanan and Tullock (1962, p. 115):

> The implication of this hypothesis suggests that the more homogenous community should adopt more inclusive rules for the making of collective decisions. However, the homogeneity characteristic affects external costs as well as decision-making costs. Thus, the community of homogenous persons is more likely to accept less restrictive rules even though it can "afford" more restrictive ones. By contrast, the community that includes sharp differences among individual citizens and groups cannot afford the decision-making costs involved in near unanimity rules for collective choice, but the very real fears of destruction of life and property from collective action will prompt the individual to refuse anything other than such rules.

Buchanan and Tullock seem to talk themselves out of a conclusion similar to ours. They suggest that more inclusive rules, like unanimity rule, might be more appropriate for both homogenous and heterogenous societies. Our results suggest that their initial intuition may have been more accurate. Everything else equal, larger k-majority rules will be optimal in homogenous societies and smaller k-majority rules will be optimal in heterogenous societies.

Finally, this analysis may provide a loose explanation for the ubiquitous use of majority rule in legislative settings. If an institutional framer is uncertain about whether individuals will typically get along and favor each other's proposals or conflict with their colleagues and oppose each other's proposals, he/she may assume that favoring and opposing proposals is equally likely, in which case, the optimal k-majority rule might be near majority rule, particularly if he/she believes a series of votes might be long and protracted. This seems to be the case in many legislative settings where policies on defense, transportation, and public welfare seem to be stepping stones in a continuous progression of decision making. With a sharply increasing decision cost function, total costs may be minimized at or near majority rule. This provides a partial justification for the wide use of majority rule without requiring a big jump discontinuity in the decision cost function.

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