Probabilistic Voting Models

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Abstract

This paper is about models of electoral competition – with an emphasis on models where there is probabilistic voting. Section 1 has (i) an example in which the voters' choices are assumed to be deterministic and (ii) an example in which the voters' choices are assumed to have probabilities that satisfy Luce's axiom of "independence from irrelevant alternatives". Section 2 has a more general model, which includes the two examples as special cases. Section 3 discusses some work that has been done on deterministic voting models. Section 4 discusses some work that has been done on probabilistic voting models.

1 Introduction

In many nations and at many levels of government, the question of who should hold various political offices is settled by voting. As a consequence, there are many public officials, and many politicians who would like to become public officials, who are concerned about the implications of their choices for the decisions that voters will make. In particular, they are concerned about which voters will vote in the next election and who these people will vote for. Two choices that elected officials and officeseeking politicians make that are especially important in this regard are: Their positions on the leading policy questions of the day and the way in which they allocate their campaign resources. As a consequence, researchers have developed mathematical models of the relation between these choices and the decisions that public officials and office-seeking politicians can expect the voters to make. These models can be used to address questions such as: In a given situation in which campaign resources have already been allocated, are there "best" policy positions that can be taken? If so, what are they? In a given situation in which policy positions have already been selected, is there a "best" campaign resource allocation that can be chosen? If so, what is it? In a given situation in which both policy positions and an allocation of resources can be selected, is there a "best" possible combination? If so, what is it? The answers are useful for social scientists who want to know what policy positions and campaign resource allocations can be expected from public officials and office-seeking politicians. They are also potentially useful for campaign strategists who advise political candidates.

2 Two Examples

Before stating the features that differ in the two examples in this section, the characteristics that they have in common will be covered: There are two candidates for a particular political office. They will be indexed by the elements in the set $C = \{1, 2\}$. There are three voters. They will be indexed by the elements in the set $N = \{1, 2, 3\}$.

Each candidate has to decide on what policy proposals he or she wants to make. The only characteristic of a candidate's policy proposals that matters to the voters is the distribution of income that they expect the policies to lead to. For each vector of policy proposals that a candidate can make, each voter has a unique distribution of income that he or she expects the policies to lead to. These expectations are the same for all three voters. As a consequence, the candidate's decision can be viewed as the choice of an income distribution. It will be assumed that each candidate can, in particular, choose any $\psi_c \in X = \{(x_1, x_2, x_3) \in R^3 : x_1 + x_2 + x_3 = 1, x_1 \geq 0.01, x_2 \geq 0.01, x_3 \geq 0.01\}$ (by a suitable choice of policy proposals).

Each particular voter cares only about his or her own income. More specifically, (for any given $i \in N$) for each pair $x, y \in X : x$ is at least as good as y for i if and only if $x_i \geq y_i$. This implies that i prefers x to y (i.e., x is at least as good as y for i, but y is not at least as good as x for i) if and only if $x_i > y_i$. It also implies that i is indifferent between x and y (i.e., x is at least as good as y for i and y is at least as good as x for i) if and only if $x_i = y_i$. Thus i's preferences on X can be *represented* by the *utility function* $U_i(x) = x_i$; for each pair $x, y \in X, U_i(x) \geq U_i(y)$ if and only if x is at least as good as y for i. In addition, assume that $U_i(x)$ measures the intensity of i's preferences. More specifically, the values assigned by $U_i(x)$ have the following interpretations: $U_i(x)/U_i(y) = \frac{1}{2}$ if and only if i likes x only half as much as he or she likes y; $U_i(x)/U_i(y) = 2$ if and only if i likes x twice as much as he or she likes y; $U_i(x)/U_i(y) = 3$ if and only if i likes x three times as much as he or she likes y; and so on. This property implies that each voter's utility function is unique up to multiplication by a positive scalar. (It is, accordingly, called a *ratio-scale utility function.*)

In the examples, $P_i^c(\psi_1, \psi_2)$ will be used to denote the probability that the individual indexed by i will vote for candidate c when $c = 1$ chooses ψ_1 and $c = 2$ chooses ψ_2. Thus [at any given $(\psi_1, \psi_2) \in X^2$] the expected vote for a given $c \in C$ can be written as

$$\mathrm{EV}^c(\psi_1, \psi_2) = \sum_{i=1}^{3} P_i^c(\psi_1, \psi_2).$$

Each candidate c is concerned solely about his or her expected plurality.

$$\mathrm{Pl}^c(\psi_1, \psi_2) = \mathrm{EV}^c(\psi_1, \psi_2) - \mathrm{EV}^k(\psi_1, \psi_2)$$

(where k is the index for the other candidate), that is, with his or her expected margin of victory (or, phrased differently, how much he or she expects to win or lose by). Furthermore, each candidate wants to maximize his or her expected plurality; in a candidate's view, the larger the expected margin of victory, the better. This implies that, for any specification of the $P_i^c(\cdot)$ functions, the decisions that the two candidates have to make can be appropriately modeled as a two-person, noncooperative game, $(X, X; \mathrm{Pl}^1, \mathrm{Pl}^2)$, in which (1) the two players are the two candidates, (2) the strategy set for each candidate is X, and (3) the payoff functions are $\mathrm{Pl}^1 : X \times X \to R^1$ and $\mathrm{Pl}^2 : X \times X \to R^1$, respectively. By the definitions of Pl^1 and Pl^2, $\mathrm{Pl}^1(\psi_1, \psi_2) + \mathrm{Pl}^2(\psi_1, \psi_2) = 0, \forall(\psi_1, \psi_2) \in X^2$. Hence the game is zero-sum.

Consider the case in which, for each $i \in N$ and each pair $(\psi_1, \psi_2) \in X^2$,

$$P_i^1(\psi_1, \psi_2) = \begin{cases} 1 & \text{if } U_i(\psi_1) > U_i(\psi_2) \\ \frac{1}{2} & \text{if } U_i(\psi_1) = U_i(\psi_2) \\ 0 & \text{if } U_i(\psi_1) < U_i(\psi_2) \end{cases}$$

and

$$P_i^2(\psi_1, \psi_2) = 1 - P_i^1(\psi_1, \psi_2).$$

The resulting game, $(X, X;\ \mathrm{Pl}^1,\ \mathrm{Pl}^2)$, has *no* Nash equilibrium. Since the game is zero-sum, this can also be phrased as: The game has *no* saddle point.

The fact that this game has no Nash equilibrium (or, equivalently, has no saddle point) can be seen quite easily. Choose any $(x,\, y) \in X^2$. Since the game is zero-sum, there are three possibilities:

$$\mathrm{Pl}^1(x,y) = \mathrm{Pl}^2(x,y) = 0, \tag{\textbf{a}}$$

$$\mathrm{Pl}^1(x,y) > 0 > \mathrm{Pl}^2(x,y), \text{ or} \tag{\textbf{b}}$$

$$\mathrm{Pl}^1(x,y) < 0 < \mathrm{Pl}^2(x,y). \tag{\textbf{c}}$$

Suppose that (**a**) or (**b**) holds. Identify a voter, i, who gets at least as much income at x as anyone else (i.e., for whom we have $x_i \geq x_j, \forall j \neq i$). Let z be the alternative in which this individual gets $z_i = 0.01$ and the others get $z_j = x_j + \frac{1}{2}(x_i - 0.01)$ (i.e., the others split the decrease in i's income). Since $x_i > 0.01$, we have $U_j(z) = z_j > x_j = U_j(x)$ for each $j \neq i$ and $U_i(z) = z_i < x_i = U_i(x)$. Therefore, $P_j^2(x,z) = 1$ and $P_j^1(x,z) = 0$ for each $j \neq i$ and, in addition, $P_i^2(x,z) = 0$ and $P_i^1(x,z) = 1$. Therefore, $\mathrm{Pl}^2(x,z) = +1 > 0 \geq \mathrm{Pl}^2(x,y)$. Therefore, if (**a**) or (**b**) holds, $(x,\, y)$ is *not* a Nash equilibrium. Similar reasoning applies when (**c**) holds.

Example 2 is a game-theoretic version of the voting paradox, the fact that when majority rule is used to make collective decisions, a society can find itself in a situation where, for each social choice that could be made, there is a feasible alternative that a majority of the voters prefer (and, hence, any particular choice can be overturned by a majority vote). More specifically, this example illustrates how easy it is for the paradox of voting (and the corresponding absence of equilibrium policies) to occur when the issues that are to be resolved involve the distribution of income in the society.

Consider the case in which the probabilities that describe the voters' choices satisfy the following version of Luce's axiom of "independence from irrelevant alternatives" (Luce [26]): For each $i \in N$ and $(\psi_1, \psi_2) \in X^2$,

$$\frac{P_i^1(\psi_1,\psi_2)}{P_i^2(\psi_1,\psi_2)} = \frac{U_i(\psi_1)}{U_i(\psi_2)}; \tag{1}$$

assume also (implicit in Example 2) that each voter is going to vote; that is, for each $i \in N$ and $(\psi_1, \psi_2) \in X^2$,

$$P_i^1(\psi_1,\psi_2) + P_i^2(\psi_1,\psi_2) = 1. \tag{2}$$

Equations (1) and (2) imply that for each $i \in N$ and $(\psi_1, \psi_2) \in X^2$,

$$\begin{aligned} P_i^1(\psi_1,\psi_2) &= \frac{U_i(\psi_1)}{U_i(\psi_1) + U_i(\psi_2)}, \\ P_i^2(\psi_1,\psi_2) &= \frac{U_i(\psi_2)}{U_i(\psi_1) + U_i(\psi_2)}. \end{aligned}$$

This time, the resulting game $(X, X;\ \mathrm{Pl}^1,\ \mathrm{Pl}^2)$ *does* have a Nash equilibrium. Since the game is zero sum, this can also be phrased as: The game *has* a saddle point.

The fact that this game has a Nash equilibrium (or equivalently, has a saddle point) can be seen quite easily. For each $i \in N$ and $(x, y) \in X^2, P_i^1(x, y) = x_i/(x_i + y_i)$. Therefore, each $P_i^1(x, y)$ is a concave function of $x = (x_1, x_2, x_3)$, and each $-P_i^1(x, y)$ is a concave function of $y = (y_1, y_2, y_3)$. This, in turn, implies that $\mathrm{EV}^1(x, y)$ is a concave function of x and $-\mathrm{EV}^1(x, y)$ is a concave function of y. Similarly, $\mathrm{EV}^2(x, y)$ is a concave function of y and $-\mathrm{EV}^2(x, y)$ is a concave function of x. Therefore, candidate 1's payoff function, $\mathrm{Pl}^1(x, y) = \mathrm{EV}^1(x, y) - \mathrm{EV}^2(x, y)$, is a concave function of x and candidate 2's payoff function, $\mathrm{Pl}^2(x, y) = \mathrm{EV}^2(x, y) - \mathrm{EV}^1(x, y)$, is a concave function of y. By a similar argument, $\mathrm{Pl}^1(x, y)$ and $\mathrm{Pl}^2(x, y)$ are continuous functions of (x, y). Finally, from its definition, X is a compact, convex subset of R^3. Hence all of the assumptions in the premise of one of the theorems that has been labeled "Nash's Theorem" are satisfied. Therefore, there is a Nash equilibrium in this example.

Where is (or are) the Nash equilibrium (or equilibria) located? This question can be answered by solving the problem: Find the $x \in X$ that maximize(s) $U_1(x) \cdot U_2(x) \cdot U_3(x) = x_1 \cdot x_2 \cdot x_3$ over the set X. There is a unique x which solves this problem: $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This implies that there is a unique Nash equilibrium in the game: $\psi_1 = \psi_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. (See Theorem 4.2 in Coughlin [7].)

3 General Model

Models similar to the two examples given above have been studied in the references listed at the end of this entry. They are all special cases of the following general model:

There are two candidates for a particular political office. They will be indexed by the elements in the set $C = \{1, 2\}$. There is a set of possible "locations," X, for the candidates. The set of possible locations is the same for both of them. A particular $x \in X$ could, for instance, specify a position on the policy issues in the election, an allocation of campaign resources, or both of these. ψ_c will be used to denote a particular location for c. There is an index set, N, for the individuals who can vote in the election. A particular $i \in N$ could be a number (e.g., if the voters are labeled as voters $1, \ldots, n$), a specification of various characteristics that a voter can have (such as location of residence, income, age, etc.), a vector that specifies the "ideal" positions on the various policy issues for a voter, or something else.

For each $i \in N$ and $c \in C$, there is a function

$$P_i^c : X \times X \to [0, 1]$$

that assigns to each $(\psi_1, \psi_2) \in X \times X$ a probability for the event "a voter randomly drawn from the individuals labeled by $i \in N$ will vote for c if candidate 1 is located at ψ_1, and candidate 2 is located at ψ_2." For each $(\psi_1, \psi_2) \in X^2, P_i^0(\psi_1, \psi_2) = 1 - P_i^1(\psi_1, \psi_2) - P_i^2(\psi_1, \psi_2)$ will be the probability for the event "a voter randomly drawn from the individuals labeled by $i \in N$ will not vote (i.e., will abstain from voting), if candidate 1 is located at ψ_1 and candidate 2 is located at ψ_2." These can be objective probabilities or they can be subjective probabilities that are believed by both of the candidates.

There is a probability distribution on N which assigns to each measurable set, $B \subseteq N$, a probability for the event "a voter randomly drawn from the individuals who can vote in the election has an index $i \in B$." These probabilities can also be either objective probabilities or subjective probabilities that are believed by both of the candidates.

Each candidate is concerned solely about his or her (a) expected vote, $\mathrm{EV}^c(\psi_1, \psi_2)$, (b) expected plurality, $\mathrm{Pl}^c(\psi_1, \psi_2)$, or (c) probability of winning $W^c(\psi_1, \psi_2)$.

4 Deterministic Models

An important special case for the general model given above is the *deterministic voting model*. This terminology comes from analyses of models in which each candidate wants to maximize his or her expected plurality (as in the examples). When [for a given $i \in N$ and $(\psi_1, \psi_2) \in X^2$] $P_i^1(\psi_1, \psi_2) = P_i^2(\psi_1, \psi_2)$, the expected vote from the individual(s) indexed by "i" is split evenly between the two candidates. When this occurs (since each candidate's objective is to maximize his or her expected plurality), the expected votes corresponding to the index i cancel each other outand, therefore, have no effect on $\mathrm{Pl}^1(\psi_1, \psi_2)$ or $\mathrm{Pl}^2(\psi_1, \psi_2)$. For instance, in Example 2, if $U_3(\psi_1) = U_3(\psi_2)$, then $P_3^1(\psi_1, \psi_2) - P_3^2(\psi_1, \psi_2) = \frac{1}{2} - \frac{1}{2} = 0$. Therefore, the expected plurality for candidate 1 when only voters 1 and 2 are counted is the same as when voters 1, 2, *and* 3 are counted. That is,

$$\begin{aligned}
&[P_1^1(\psi_1, \psi_2) + P_2^1(\psi_1, \psi_2)] \\
&\qquad - [P_1^2(\psi_1, \psi_2) + P_2^2(\psi_1, \psi_2)] \\
&\quad = [P_1^1(\psi_1, \psi_2) + P_2^1(\psi_1, \psi_2) + P_3^1(\psi_1, \psi_2)] \\
&\qquad - [P_1^2(\psi_1, \psi_2) + P_2^2(\psi_1, \psi_2) + P_3^2(\psi_1, \psi_2)].
\end{aligned}$$

Similarly, the expected plurality for candidate 2 when only voters 1 and 2 are counted is the same as when voters 1, 2, *and* 3 are counted.

From the preceding observations it is thus clear that at any given (ψ_1, ψ_2), the only voter indices that matter (to expected plurality maximizing candidates) are ones with $P_i^1(\psi_1, \psi_2) \neq P_i^2(\psi_1, \psi_2)$. When, in fact, $P_i^1(\psi_1, \psi_2) = 1$ or $P_i^2(\psi_1, \psi_2) = 1$ at a given $i \in N$ and $(\psi_1, \psi_2) \in X^2$, one of two things must be true: (a) there is one voter with the index i and the candidates believe that his or her decision will be completely determined once they choose the strategies ψ_1 and ψ_2, respectively, *or* (b) there is more than one voter with the index i and the decisions made by all these voters will be completely determined (and the same) once the candidates choose the strategies ψ_1 and ψ_2, respectively. Because of this, any model which satisfies the assumptions of the general voting model given above *and* is such that, at each $(\psi_1, \psi_2) \in X^2$ and $i \in N$ where $P_i^1(\psi_1, \psi_2) \neq P_i^2(\psi_1, \psi_2)$, either (i) $P_i^1(\psi_1, \psi_2) = 1$ or (ii) $P_i^2(\psi_1, \psi_2) = 1$, is called a "deterministic voting model." Example 2, for instance, is such a model, whereas Example 2 is not. The defining characteristic of a deterministic voting model can be restated as: For each $(\psi_1, \psi_2) \in X^2$ and $i \in N$, either the expected votes corresponding to the index i cancel each other out (and, therefore, have no effect on the candidates' expected pluralities) *or* the decision(s) of the voter(s) corresponding to the index i will be completely determined (and identical when candidate 1 chooses ψ_1 and candidate 2 chooses ψ_2. A third way of stating this characteristic is: For each $(\psi_1, \psi_2) \in X^2$ and $i \in N$, (a) $P_i^1(\psi_1, \psi_2) = 1$, (b) $P_i^2(\psi_1, \psi_2) = 1$, *or* (c) $P_i^1(\psi_1, \psi_2) = P_i^2(\psi_1, \psi_2) = \frac{1}{2}(1 - P_i^0(\psi_1, \psi_2))$.

The deterministic voting models that have received the most attention are ones in which (1) each index corresponds to one voter, (2) each voter, i, has a utility function, $U_i(x)$, and (3a) for each voter, i, and each $(\psi_1, \psi_2) \in X^2$,

$$\begin{aligned} P_i^1(\psi_1, \psi_2) &= \begin{cases} 1 & \text{if } U_i(\psi_1) > U_i(\psi_2), \\ \frac{1}{2} & \text{if } U_i(\psi_1) = U_i(\psi_2), \\ 0 & \text{if } U_i(\psi_1) < U_i(\psi_2), \end{cases} \\ P_i^2(\psi_1, \psi_2) &= 1 - P_i^1(\psi_1, \psi_2) \end{aligned}$$

(as in Example 2) or (3b) for each voter, i, and each $(\psi_1, \psi_2) \in X^2$

$$\begin{aligned} P_i^0(\psi_1, \psi_2) &= \begin{cases} 1 & \text{if } U_i(\psi_1) = U_i(\psi_2), \\ 0 & \text{if } U_i(\psi_1) \neq U_i(\psi_2), \end{cases} \\ P_i^1(\psi_1, \psi_2) &= \begin{cases} 1 & \text{if } U_i(\psi_1) > U_i^0(\psi_2), \\ 0 & \text{if } U_i(\psi_1) \leq U_i(\psi_2), \end{cases} \\ P_i^2(\psi_1, \psi_2) &= 1 - P_i^1(\psi_1, \psi_2) - P_i^0(\psi_1, \psi_2). \end{aligned}$$

An exception is McKelvey [27], where it is assumed that (a) for each index, α, all of the voters who are labeled by this index have the same utility function, $U_\alpha(x)$, and (b) for each index α and each $(\psi_1, \psi_2) \in X^2$,

$$\begin{gathered} (P_\alpha^0(\psi_1, \psi_2), P_\alpha^1(\psi_1, \psi_2), P_\alpha^2(\psi_1, \psi_2)) \\ \in \{(1,0,0), (0,1,0), (0,0,1)\}, \\ P_\alpha^1(\psi_1, \psi_2) = 1 \Rightarrow U_\alpha(\psi_1) > U_\alpha(\psi_2), \\ P_\alpha^2(\psi_1, \psi_2) = 1 \Rightarrow U_\alpha(\psi_1) < U_\alpha(\psi_2). \end{gathered}$$

This formulation provides a deterministic voting model in which abstentions can occur more frequently than just when $U_\alpha(\psi_1) = U_\alpha(\psi_2)$.

The seminal work on deterministic voting models was done by Hotelling [22], Downs [15], Black [2], and Davis and Hinich [10,11]. Surveys of results that have been derived in these and other analyses [using either $\mathrm{Pl}^c(\psi_1, \psi_2)$ or $W^c(\psi_1, \psi_2)$ as the objective functions for the candidates] can be found in Mueller [28], Duggan [16], and Coughlin [8].

5 Probabilistic Models

Any model which satisfies the assumptions of the general voting model given above, but is *not* a deterministic voting model, is called a *probabilistic voting model*. Example 2, for instance, is such a model. The terminology reflects the fact that in any such model, there is at least one pair of strategies $(\psi_1, \psi_2) \in X^2$ at which there is at least one index that matters (to expected plurality maximizing candidates) where the candidates' beliefs about the voter(s) corresponding to the index are probabilistic in a nontrivial way; the random variable that describes them is non-degenerate. The defining characteristic for these models can be restated as: There is at least one $(\psi_1, \psi_2) \in X^2$ and $i \in N$ where $P_i^1(\psi_1, \psi_2) \neq P_i^2(\psi_1, \psi_2)$ *and* $0 < P_i^1(\psi_1, \psi_2) < 1$ or $0 < P_i^2(\psi_1, \psi_2) < 1$ (or both).

Any model that satisfies all the assumptions of the general voting model given above will be a probabilistic voting model if there is an index i that corresponds to two or more voters and at least one pair of possible locations $(\psi_1, \psi_2) \in X^2$ such that the choices of the voters corresponding to index i are completely determined when candidate 1 chooses ψ_1 and candidate 2 chooses ψ_2, *but* (a) they will not all make the same choice, and (b) those who will vote are not split evenly between the candidates.

Indeed, almost any deterministic voting model can be converted into a probabilistic voting model of this sort by appropriately regrouping the indices in the deterministic voting model into indices for the probabilistic voting model. For instance, let all three voters in Example 2 have the same index i. Then, for the strategies $\psi_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\psi_2 = (0, \frac{1}{2}, \frac{1}{2})$, we have $P_i^1(\psi_1, \psi_2) = \frac{1}{3}$ and $P_i^2(\psi_1, \psi_2) = \frac{2}{3}$. The resulting model is a probabilistic voting model. This approach is the basis for models analyzed in McKelvey [27]. A voting model that satisfies all the assumptions of the general voting model given above can, alternatively, be a probabilistic voting model if there is at least one voter whose choice of whether to vote and/or which candidate to vote for (if he or she votes) is probabilistic in nature, as in Example 2. This approach is the basis for models analyzed in Brams [3], Brams and Davis [4], Comaner [5], Coughlin and Nitzan [9], Davis and Hinich [12], Hinich [18], Hinich and Ordeshook [19], Hinich . [20,21], and Lake [23]. A voting model that satisfies all the assumptions given above could be a probabilistic voting model because the candidates are uncertain about the choices that voters will make and use subjective probabilities to summarize their expectations about these choices. This is the basis for the model analyzed in Ledyard [24,25]. Analyses that apply to models in which any of these three interpretations can arise have been carried out in Coughlin [7].

The most closely scrutinized probabilistic voting models can be grouped into three basic categories. The first consists of models in which the description of the candidates' expectations about the voters' decisions to vote (or abstain) are probabilistic, but the description of their expectations about the choices that the voters make between the candidates are deterministic (i.e., they believe that, for each $(\psi_1, \psi_2) \in X \times X$ and $i \in N$, when candidate 1 chooses ψ_1 and candidate 2 chooses ψ_2, the voters corresponding to i who vote will all vote for the same candidate and the candidate who so benefits is completely determined). This category includes models in Davis and Hinich [12], Hinich and Ordeshook [19], Hinich . [20,21], and McKelvey [27]. The second category consists of models in which there are no abstentions, but the description of the candidates' expectations about the choices that the voters will make in choosing between them is probabilistic: (a) $P_i^0(\psi_1, \psi_2) = 0, \forall(\psi_1, \psi_2) \in X \times X, \forall i \in N$ and (b) $\exists i \in N$ and $(\psi_1, \psi_2) \in X \times X$, where $P_i^1(\psi_1, \psi_2) \neq \frac{1}{2}$. This category includes models in Brams [3], Brams and Davis [4], Hinich [18], Coughlin and Nitzan [9], and Coughlin [7]. The third category consists of models in which the description of the candidates' expectations both about voter abstentions and about the choices that the voters will make in choosing between them are probabilistic. This category includes models in Denzau and Kats [14], Coughlin [6], and Ledyard [24,25].

For further discussion on these topics, see Gill and Gainous [17], Mueller [28], Austen-Smith and Banks [1], Duggan [16], or Coughlin [8].

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