

Representative Democracies: can we avoid vote swapping?

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1 introduction

This paper examines the problem of *manipulation by vote swapping* in constitutions equipped with two step elections, such as Great-Britain, Canada or the US. In these constitutions, the country is divided into jurisdictions in which elections are held at the first step in order to choose the jurisdictional winners. In the second step, the government is appointed by an aggregation procedure over all the jurisdictional winners. It is a well-know fact that these constitutions, with usual voting rules, are sensitive to the partition of the electors.

In particular, one problem could come from vote swapping. Vote swapping is an informal agreement where two voters from different jurisdictions and parties trade votes to get representatives from their party elected while at the same time blocking out an unwanted third party from being elected to office.

Many websites enable to swap votes (give examples). They claim that they have reversed the result of some elections (Canada for instance). The problem became big and in 2007 in the US, the court of appeal has been asked to give a ruling on vote swapping. The court concluded that "vote swapping mechanisms as well as the communication and vote swaps that websites enabled were constitutionally protected". The problem becomes real and raises the issue of manipulability of such elections. Note that manipulation by vote swapping is different from manipulation la Gibbard-Satterthwaite which refers to a misrepresentation of preferences. The question raised in this paper is to know whether it is possible to design some electoral rules which are immune to vote swaps.

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What we do: we fix the size of every jurisdiction (possibly all different) and we allow voters to swap their votes. Votes are collected and aggregated. Local winners are designated by some local voting rule (possibly all different), and all the local winners are aggregated by another voting rule, so as to elect the federal winner.

When imposing anonymity on the local voting rules, vote swapping is logically equivalent to *gerrymandering* when the size of jurisdictions is fixed. Gerrymandering is a term that describes the deliberate rearrangement of the boundaries of electoral districts to influence the outcome of elections. The original gerrymander was created in 1812 by Massachusetts governor Elbridge **Gerry**, who crafted a district for political purposes that looked like a sala-**mander**. The purpose of gerrymandering is to concentrate opposition votes into a few districts or gain more seats for the majority in surrounding districts (called packing), or to diffuse minority strength across many districts (called dilution). The literature has examined the problem of gerrymandering (see Bervoets and Merlin(2011) or Chambers (2008)) but assumes that the size of the jurisdictions can be changed in order to manipulate. This is an obvious weakness of the analysis as in real life elections the size of every jurisdiction is fixed. In this paper on vote swapping, this problem is fixed as the size of the jurisdictions cannot change.

What we obtain: the set of voting rules which are swap-proof has three very undesirable properties: either the voters are all pivotal against unanimity (i.e. whenever everyone votes for the same candidate except for one voter, this unique voter can decide on the outcome of the election), or the voting rules are such that candidates can be elected even though they received no votes (typically b is elected when everyone votes for a) or the voting rule is an exogenous code imposing the elected candidate regardless of the votes.

This result resonates as an impossibility to guarantee that elections can be swap-proof.

2 The general framework

2.1 Notations and Definitions

Let A be a finite and fixed set of candidates, $N = \{1, \dots, n\}$ the fixed set of voters, with $n \geq 3$, and $J = \{J_1, \dots, J_m\}$ the set of jurisdictions, with $m \geq 2$. We assume that $n > m$. Let n_j be the size of jurisdiction J_j , i.e. the number of voters in J_j . Of course $\sum_j n_j = n$.

Let σ be a partition function from N to $\{1, \dots, m\}$. Formally, for all $i \in N$, $\sigma(i) = j \Leftrightarrow i \in J_j$, with $\bigcup_{j=1, m} J_j = N$ and $J_j \cap J_k = \emptyset$ when $j \neq k$. In what follows, we consider partition functions in Σ^f , defined as the set of all partitions such that $\sigma^{-1}(j) \neq \emptyset$ and $Card\{i; \sigma(i) = j\} = n_j$ for all $j \in \{1, \dots, m\}$. There is no empty jurisdiction, there is at least one jurisdiction with strictly more than one voter and the size of the jurisdictions is fixed.

If we consider vote swapping between two voters as equivalent to home swapping, we need to define σ_{ij} as the partition of society such that $\sigma_{ij}(j) = \sigma(i)$ and $\sigma_{ij}(i) = \sigma(j)$ while $\sigma_{ij}(k) = \sigma(k)$ otherwise.

Voters vote for one unique candidate in the jurisdiction they reside in and votes are taken as given. We assume that when voters swap their votes, there is no trust issue, voters are honest.

$\pi \in A^n$ denotes a *vote profile*. Typically, a vote profile is identified with a vector of a 's, b 's, \dots where the i^{th} coordinate indicates voter i 's vote. $\pi|_i$ denotes voter i 's vote and for any subset S of N , we denote by $\pi|_S$ the restriction of π to S . When vote swapping between two individuals i and j occurs, we consider the vote profile π_{ij} such that $\pi_{ij}|_{N \setminus \{i,j\}} = \pi|_{N \setminus \{i,j\}}$ and $\pi_{ij}|_i = \pi|_j$, $\pi_{ij}|_j = \pi|_i$.

Once the vote profile is given, the winner in jurisdiction J_j is chosen via the social choice function f_j :

$$\begin{aligned} f_j : \Sigma^f \times A^n &\rightarrow A \\ (\sigma, \pi) &\rightarrow z \in A \end{aligned}$$

We impose the following very mild condition on the functions $\{f_j\}_{j=1,m}$:

Jurisdiction Sovereignty

If $[\sigma(i) = j \Leftrightarrow \sigma'(i) = j]$ and $[\pi|_{J_j} = \pi'|_{J_j}]$ then $f_j(\sigma, \pi) = f_j(\sigma', \pi')$ for all j

In words, the result of an election in jurisdiction J_j is independent of what happens in other jurisdictions. The set of all social choice functions satisfying Jurisdiction Sovereignty is denoted by \mathcal{F} .

The m jurisdictional winners constitute a jurisdictional vote profile $\Pi \in A^m$, called a *federal profile*. The federation then appoints a federal winner using the federal social choice function g defined as follows:

$$\begin{aligned} g : & & A^m &\rightarrow A \\ \Pi = (z_1, \dots, z_m) &\rightarrow & z \in A \end{aligned}$$

Function g 's domain is restricted to the set of all jurisdictional elected candidates. It does not include any other type of information such as the number of voters who elected each candidate, the margin of victory, etc.

A *federal constitution* is given by a $(m + 1)$ -tuple $C = (g, f_1, \dots, f_m)$, with $f_j \in \mathcal{F}$ for all j . The federal winner of the election will be denoted by:

$$g(f_1(\sigma, \pi), \dots, f_m(\sigma, \pi)) = g(f(\sigma, \pi))$$

Although functions f_j and g are social choice functions, the combination of these in a two steps procedure does not generate a social choice function.

2.2 Properties

We define properties we will impose on the different social choice functions.

Minimal Representativity (MinRep)

For all $j \in J, \exists i \in N, \sigma(i) = j$, such that $f_j(\sigma, \pi) = \pi|_i$

In order to be elected in J_j , a candidate must have received at least one vote in J_j . This property implies among others the following:

Local Unanimity

If $\pi|_i = \{z\}$ for all $i \in J_j$ then $f_j(\sigma, \pi) = z$

A candidate should be chosen in his jurisdiction whenever he receives the votes of all voters.

Note that both Minimal Representativity and Local Unanimity are silent about what happens at the federal level.

Swap-proofness (SwPr)

For all $\pi \in A^n$, all $\sigma \in \Sigma^f$,

$$g(f(\sigma, \pi)) = g(f(\sigma, \pi_{ij})) \text{ for all } i, j \in N$$

Gerrymander-proofness with fixed jurisdiction size (GPf)

For all $\pi \in A^n$, $g(f(\sigma, \pi)) = g(f(\sigma', \pi))$ for all $\sigma, \sigma' \in \Sigma^f$

Proposition 1 *A constitution C is gerrymander-proof with fixed jurisdiction size if and only if it is swap-proof with anonymous local voting rules.*

Proof of Proposition 1: to be provided

Pivotal Voters It seems reasonable to require that constitutions do not concentrate the decision in the hands of a unique voter. We express part of this idea with the following definitions, stating that one voter alone cannot overrule unanimity.

Let the *Unanimous vote profile* π_z be defined as the vote profile such that $\pi|_i = z$ for all i in N , z in A .

Pivotal voter in π_z : Voter i is called pivotal in π_z with $z \in A$ if for all σ there exists $y \in A$, $z \neq y$ and π such that $\pi|_i = y$ and $\pi|_h = z$, $h \neq i$ and $g(f(\sigma, \pi_z)) \neq g(f(\sigma, \pi))$.

Pivotal voter: Voter i is called pivotal if there exists an alternative $z \in A$ such that i is pivotal in π_z .

A pivotal voter is thus pivotal for at least one unanimous vote profile. Note that a pivotal voter i does not have the power to impose his choice on society, he can only change the winner on his own by voting for y at some unanimous profile π_z (but his choice could be a third candidate). Furthermore, being pivotal is different from having a veto power, as pivotal voters have power only against the unanimity.

The following axiom states that there should be no pivotal voters

No Pivotal Voters (NoPiv) In C there is no pivotal voter

2.3 Theorem 1

Theorem 1 *Let $A = \{a, b\}$. Then C satisfies NoPiv, MinRep and SwPr if and only if it is the Exogenous Code.*

Proof: the Exogenous Code trivially satisfies NoPiv, MinRep and SwPr.

Consider any partition σ and the individual profile $\pi_0 = a\dots a$. By MinRep, $f_j(\pi_0, \sigma) = a$ for all j . Call z the federal winner associated to the federal profile $\Pi = a\dots a$ (i.e. $g(f(\pi_0, \sigma)) = z$).

Let $t_j(\sigma, a, b)$ be the minimal number of votes for b in J_j such that the winner in J_j is b , whenever all the other votes are for candidate a . Of course, as the f_j 's might not be anonymous, we could have $t_j(\sigma, a, b) \neq t_j(\sigma', a, b)$. Given MinRep, $0 < t_j(\sigma, a, b) \leq n_j$. Let $t_j^*(a, b)$ be the minimal value of $t_j(\sigma, a, b)$ for any σ . Then let S_1 be the set of j 's such that $t_j^*(a, b) = 1$ and S_+ be the set of all j 's such that $2 \leq t_j^*(a, b) \leq n_j$.

For any federal profile Π , let $P(\Pi)$ be the number of jurisdictions J_j which have elected b . Obviously, $P(\Pi)$ ranges from 0 to m .

We need to show that if C satisfies NoPiv, MinRep and SwPr then $g(\Pi) = z$ for any Π such that $0 \leq P(\Pi) \leq m$. We do this by induction over natural numbers, by first showing that $[g(\Pi) = z \text{ when } P(\Pi) = 0]$ implies that $[g(\Pi) = z \text{ for all } \Pi \text{ such that } P(\Pi) = 1]$. We will next assume that $[g(\Pi) = z \text{ whenever } P(\Pi) \leq k]$ and show that this implies that $[g(\Pi) = z \text{ whenever } P(\Pi) = k + 1]$.

Starting the induction: assume $g(\Pi) = z$ when $P(\Pi) = 0$. For any jurisdiction J_j such that $j \in S_1$ the condition NoPiv implies that $g(a, \dots, b, \dots, a) = z$. This is enough to conclude for the jurisdictions in S_1 , but not for the others as g is not necessarily anonymous. Now, for jurisdictions belonging to S_+ we consider two cases:

Case 1: $Card(S_+) = 1$ and $Card(S_1) \geq 1$. Let j belong to S_+ and k belong to S_1 . Then the individual vote profile such that everyone votes for a except for one individual who votes for b in J_k . That individual is chosen such that $f_k(a, \dots, b, a, \dots) = b$. Then $g(a, \dots, \underbrace{a}_{J_j}, \dots, \underbrace{b}_{J_k}, \dots, a)$ gives z as the winner by using NoPiv. Changing

$\pi|_{J_j}$ to $(a\dots a \underbrace{b\dots b}_{t_j(\sigma, a, b)-1})$ does not change the federal winner as the winner in J_j is

still a and by swapping the b vote in J_k for the appropriate vote for a in J_j we get $g(a, \dots, \underbrace{b}_{J_j}, \dots, \underbrace{a}_{J_k}, \dots, a) = z$ by SwPr.

Case 2: $Card(S_+) \geq 2$. Let j and k belong to S_+ , consider the partition σ and the vote profile π such that $\pi|_{J_l} = a\dots a$ for all $l \neq j, k$, $\pi|_{J_j} = (a\dots a \underbrace{b\dots b}_{t_j(\sigma, a, b)-1})$ and

$\pi|_{J_k} = (a \dots a \underbrace{b \dots b}_{t_k(\sigma, a, b) - 1})$. Both J_j and J_k elect a and every J_l elects a as well. Hence the winner is z . By swapping one appropriate vote for b from J_k with one appropriate vote for a from J_j the federal profile becomes $g(a, \dots, \underbrace{b}_{J_j}, \dots, a)$ and by SwPr the winner is still z .

Hence whenever the federal profile contains exactly one b the winner is always z . We can turn to the induction hypothesis: assume $[g(\Pi) = z \text{ whenever } P(\Pi) \leq k, k < m]$. We will consider without loss of generality that it is the k first jurisdictions (J_1 to J_k) which have switched their vote to b . Hence $g(\underbrace{b, \dots, b}_{J_1 \text{ to } J_k}, a, \dots, a) = z$ and this can be

obtained by the vote profile π such that $\pi|_{J_j} = \underbrace{a \dots a}_{n_j - t_j} \underbrace{b \dots b}_{t_j}$ for $1 \leq j \leq k$ and $\pi|_{J_j} = \underbrace{a \dots a}_{n_j}$

for all $j \geq k + 1$.

Again we have two cases.

Case 1: there is an $l, 1 \leq l \leq k$, such that $t_l < n_l$. Consider then the profile π such that $\pi|_{J_j} = \underbrace{a \dots a}_{n_j - t_j} \underbrace{b \dots b}_{t_j}$ for all $j \leq k, j \neq l$, and $\pi|_{J_l} = \underbrace{a \dots a}_{n_l - t_l - 1} \underbrace{b \dots b}_{t_l + 1}$ and $\pi|_{J_{k'}} = \underbrace{a \dots a}_{n_{k'} - t_{k'} + 1} \underbrace{b \dots b}_{t_{k'} - 1}$ for all $k' > k$. This profile generates either the federal profile $(b, \dots, \underbrace{a}_{J_l}, \dots, b, a, \dots, a)$

or the federal profile $(b, \dots, \underbrace{b}_{J_l}, \dots, b, a, \dots, a)$ depending on the winner in J_l . But in

both cases the federal profile is such that $P(\Pi) \leq k$ so that the winner is z . Consider now swapping a vote for b in J_l with a vote for a in $J_{k'}$ for any $k' > k$. This gives the federal profile $(b, \dots, b, a, \dots, \underbrace{b}_{J_{k'}}, \dots, a)$ and by SwPr this gives z as the winner, although

Π is such that $P(\Pi) = k + 1$.

Case 2: for all $l, 1 \leq l \leq k, t_l = n_l$. We need to distinguish three subcases.

Case 2a: if $t_j < n_j$ for a $j > k$. Then consider π such that $\pi|_{J_l} = \underbrace{b \dots b}_{n_l = t_l}$ for all $l \leq k - 1$,

and $\pi|_{J_k} = a \underbrace{b \dots b}_{n_k - 1}$ and $\pi|_{J_j} = \underbrace{a \dots a}_{n_j - t_j - 1} \underbrace{b \dots b}_{t_j + 1}$. This profile generates either the federal profile

$(\underbrace{b, \dots, b}_{J_1 \text{ to } J_{k-1}}, \underbrace{a}_{J_k}, \dots, \underbrace{a}_{J_j}, \dots, a)$ or the federal profile $(\underbrace{b, \dots, b}_{J_1 \text{ to } J_{k-1}}, \underbrace{a}_{J_k}, \dots, \underbrace{b}_{J_j}, \dots, a)$ depend-

ing on the winner in J_j . But in both cases the federal profile is such that $P(\Pi) \leq k$ so that the winner is z .

Now switch one b in J_j with the unique a in J_k in order to get the federal profile $(\underbrace{b, \dots, b}_{J_1 \text{ to } J_k}, a, \dots, \underbrace{b}_{J_j}, \dots, a)$. By SwPr the winner is z .

Case 2b: if $t_j = n_j$ for a $j > k$ and $k < m - 1$, i.e. we are not at the last step of the induction. Consider the profile such that $\pi|_{J_l} = b\dots b$ for $1 \leq l \leq k$, $\pi|_{J_j} = a \underbrace{b\dots b}_{n_j-1}$ and $\pi|_{J_{j'}} = \underbrace{a\dots a}_{n_{j'}-1} b$ for a $j' > k, j' \neq j$. Then the federal profile is either

$(\underbrace{b, \dots, b}_{J_1 \text{ to } J_k}, a, \dots, \underbrace{a}_{J_j}, \dots, \underbrace{a}_{J_{j'}}, \dots, a)$ and the winner is z or it is $(\underbrace{b, \dots, b}_{J_1 \text{ to } J_k}, a, \dots, \underbrace{a}_{J_j}, \dots, \underbrace{b}_{J_{j'}}, \dots, a)$ if $t_{j'} = 1$ and then the winner is also z according to case 3a. Swapping the b in $J_{j'}$ with an a in J_j gives the federal profile $(\underbrace{b, \dots, b}_{J_1 \text{ to } J_k}, a, \dots, \underbrace{b}_{J_j}, \dots, \underbrace{a}_{J_{j'}}, \dots, a)$ and by SwPr

the winner is z .

Case 2c: if $t_j = n_j$ for a $j > k$ and $k = m - 1$. In that case we know that the profile $\pi|_{J_l} = b\dots b$ for all $1 \leq l \leq m - 1$ and $\pi|_{J_j} = a \underbrace{b\dots b}_{n_j-1}$ generates the federal profile $(\underbrace{b, \dots, b}_{J_1 \text{ to } J_{m-1}}, a)$ and this gives z as the winner (as $m - 1 = k$). By NoPiv, changing the unique a vote into a b vote cannot change the result so that (b, \dots, b) gives also z as the winner.

Q.E.D.

2.4 Theorem 2

Theorem 2 *Let A be a finite set of candidates (with possibly $f_j \neq f_k$ and $n_j \neq n_k$). Then C satisfies NoPiv, MinRep and SwPr if and only if it is the Exogenous Code.*

Proof 1 *This proof uses lemma 1 and 2 stated and proved below.*

We proceed by induction over natural numbers. Let $X(\Pi)$ be the set of all candidates which are represented in the federal profile Π . First we show that for any federal profile Π such that $\text{Card}(X(\Pi)) = 2$, the winner is always the same. Theorem 1bis tells us that when $X(\Pi) = \{a, b\}$ the winner is some option z , whatever Π . We need to show that the winner is the same z when $\text{Card}(X(\Pi)) = 2$, and for instance $X(\Pi) = \{c, d\}$.

As z is the winner in case of an unanimous profile for a , generating the federal profile $(a\dots a)$, whether the second candidate is b or c does not change the logic of Theorem 1bis. Therefore z wins any election when $X(\Pi) = \{a, c\}$. But this implies that the unanimous profile for c , generating the federal profile $(c\dots c)$ gives z as the winner. Therefore, z must also be the winner when $X(\Pi) = \{c, d\}$ by applying the reasoning of

Theorem 1bis with the pair $\{c, d\}$ instead of $\{a, b\}$.

Next we use the following induction hypothesis: the winner is z whenever $\text{Card}(X(\Pi)) < s$. We will show that this implies that the winner is still z whenever $\text{Card}(X(\Pi)) = s$, with $s < m$ (the case $s = m$ will be treated separately).

Let the individual vote profile π and the partition σ be such that $\pi|_{J_1} = \underbrace{a_1 \dots a_1}_{n_1-1} a_2$, $\pi|_{J_2} = a_1 \underbrace{a_2 \dots a_2}_{n_2-1}$, while all other jurisdictions are unanimous for candidates a_3 to a_s .

By *MinRep*, $f_1(\pi, \sigma) = a_1$ or a_2 and $f_2(\pi, \sigma) = a_1$ or a_2 . There are now three cases.

Case 1: $f_1(\pi, \sigma) = f_2(\pi, \sigma)$. In that case the federal profile Π associated to π and σ contains only $s - 1$ different alternatives and by the induction hypothesis the winner has to be z . Now by swapping the a_2 vote in J_1 for the a_1 vote in J_2 , *MinRep* implies that the winner in J_1 is a_1 and the winner in J_2 is a_2 , hence generating a federal profile Π such that $\text{Card}(X(\Pi)) = s$, while *SwPr* implies that the winner is z .

Case 2: $f_1(\pi, \sigma) = a_1$ and $f_2(\pi, \sigma) = a_2$. Then consider the vote profile π' such that $\pi|_{J_j} = \pi'|_{J_j}$ for all $j \neq 1$ and $\pi'|_{J_1} = \underbrace{a_1 \dots a_1}_{n_1-t_1} \underbrace{a_2 \dots a_2}_{t_1}$. Thus $f_1(\pi', \sigma) = a_2$ and $f_2(\pi', \sigma) = a_2$, hence generating a federal profile Π such that $\text{Card}(X(\Pi)) = s - 1$. The winner is z by the induction hypothesis. Now, by swapping one vote for a_2 in J_1 with the unique vote for a_1 in J_2 we get a_1 as the winner in J_1 and a_2 as the winner in J_2 and hence a federal profile Π such that $\text{Card}(X(\Pi)) = s$, while *SwPr* implies that the winner is z .

Case 3: $f_1(\pi, \sigma) = a_2$ and $f_2(\pi, \sigma) = a_1$. Consider in J_1 and J_2 the sequence of vote profiles, starting from $a_1 \dots a_1$, and switching individuals one by one to a_2 until reaching $a_2 \dots a_2$. There are two subcases. Along this sequence, either J_1 or J_2 are such that the winner is the same for two consecutive steps of the sequence, i.e. for $j = 1$ $j = 2$, $f_j(\underbrace{a_1 \dots a_1}_k \underbrace{a_2 \dots a_2}_{n_j-k}, \sigma) = f_j(\underbrace{a_1 \dots a_1}_{k-1} \underbrace{a_2 \dots a_2}_{n_j-k+1}, \sigma)$ or the winner in both J_1 and J_2 changes every time one individual switches his vote from a_1 to a_2 .

Case 3a: Assume that there is such a sequence in J_1 (or in J_2), i.e. there is a k such that $f_1(\underbrace{a_1 \dots a_1}_k \underbrace{a_2 \dots a_2}_{n_1-k}, \sigma) = f_1(\underbrace{a_1 \dots a_1}_{k-1} \underbrace{a_2 \dots a_2}_{n_1-k+1}, \sigma)$. If the local winner in that case is a_1 , then consider π' such that $\pi'|_{J_1} = \underbrace{a_1 \dots a_1}_{k-1} \underbrace{a_2 \dots a_2}_{n_1-k+1}$, and $\pi'|_{J_j} = \pi|_{J_j}$ when $j \neq 1$.

The local winners in J_1 and J_2 are a_1 and hence the federal profile contains only $s - 1$ candidates. The federal winner is thus z . By swapping one vote for a_2 in J_1 for the unique vote for a_1 in J_2 , we get respectively a_1 in J_1 (by hypothesis) and a_2 in J_2 (by

MinRep) and hence a federal profile with s candidates. The winner is still z by SwPr. If the local winner in that case is a_2 , then consider π'' such that $\pi''|_{J_1} = \underbrace{a_1 \dots a_1}_k \underbrace{a_2 \dots a_2}_{n_1-k}$, $\pi''|_{J_2} = \underbrace{a_2 \dots a_2}_{n_2}$ and $\pi''|_{J_j} = \pi|_{J_j}$ for $j \neq 1, 2$. The local winners are respectively a_2 and a_2 and hence there are only $s - 1$ different local winners. The federal winner is thus z . By swapping one vote for a_1 in J_1 with one vote for a_2 in J_2 we get respectively $\pi''|_{J_1} = \underbrace{a_1 \dots a_1}_{k-1} \underbrace{a_2 \dots a_2}_{n_1-k+1}$ which gives a_2 as the local winner and $\pi''|_{J_2} = \underbrace{a_2 \dots a_2}_{n_2-1} b = \pi|_{J_2}$ which gives a_1 as the local winner. By SwPr the winner is z .

Case 3b: the winner in both J_1 and J_2 changes every time one individual switches his vote from a_1 to a_2 . By MinRep, this is only possible if the size of the jurisdictions J_1 and J_2 are odd as $f_j(a_1 \dots a_1, \sigma) = a_1$ and $f_j(a_2 \dots a_2) = a_2$ and if all jurisdictions elect the candidate who receives an odd number of votes, whenever the vote profile contains only a_1 and a_2 . Consider then the profile such that $\pi^*|_{J_1} = \underbrace{a_1 \dots a_1}_{k \text{ odd}} \underbrace{a_2 \dots a_2}_{n_1-k}$, $\pi^*|_{J_2} = \underbrace{a_1 \dots a_1}_{k' \text{ odd}} \underbrace{a_2 \dots a_2}_{n_2-k'}$, $\pi^*|_{J_3} = a_1 \dots a_1$ and $\pi^*|_{J_j} = \pi|_{J_j}$ for all $j > 3$. The associated federal profile is thus given by $\Pi^* = a_1, a_1, a_1, a_3, \dots, a_s$ which contains $s - 1$ alternatives and thus gives z as the winner.

By swapping an a_2 in J_1 for an a_1 in J_2 changes both parities and produces the federal profile $\Pi' = a_2, a_2, a_1, a_3, \dots, a_s$ which contains exactly s alternatives. By SwPr the winner should be z .

What remains to be shown is that the induction is also correct when $s = m$. All the previous cases (case 1, case 2 and case 3a) still apply, it is only the case 3b which cannot be solved. Case 3b when $s = m$ implies that in every J_j and for any two candidates a and b , then $f_j(\underbrace{a \dots a}_k \underbrace{b \dots b}_{n_j-k}, \sigma) \neq f_j(\underbrace{a \dots a}_{k-1} \underbrace{b \dots b}_{n_j-k+1}, \sigma)$ for all k , i.e. the winner changes ev-

ery time one individual switches his vote from one candidate to the other whenever the profile in any jurisdiction contains two candidates. Moreover, by MinRep, this implies that every n_j is odd and that the winner in every jurisdiction is the candidate receiving an odd number of votes. Then, consider the following individual profile π^+ such that $\pi^+|_{J_1} = a_1 \dots a_1 a_2 a_3$, $\pi^+|_{J_2} = a_2 \dots a_2 a_1$, $\pi^+|_{J_3} = a_3 \dots a_3 a_1$, and $\pi^+|_{J_j} = \pi|_{J_j}$ if $j > 3$.

The local winner in J_1 is unknown, call him f_1 , but a_1 is the winner in both J_2 and J_3 as he is the candidate who receives an odd number of votes. In the other jurisdictions the winners are a_4 to a_m . This implies that the federal profile contains $s - 2$ or $s - 1$ different alternatives, according to the value of f_1 . The winner is thus z . Now, swap the unique vote for a_1 in J_3 for the unique vote for a_3 in J_1 . In this new profile, the winner in J_1 is a_2 because there are only two candidates represented and a_2 receives an odd number of votes. In J_2 the winner is still a_1 while now in J_3 the winner is a_3 by MinRep. We

thus have a federal profile with exactly $s(=m)$ alternatives and by SwPr the winner is z .

Q.E.D

Lemma 1 *Let A be a finite set of candidates and let the functions f_j be anonymous (with possibly $f_j \neq f_k$ and $n_j \neq n_k$). If C satisfies NoPiv, MinRep and SwPr then g satisfies Federal Anonymity*

Consider any federal profile Π and another profile Π' such that $\Pi'|_k = \Pi|_k$ for $k \neq i, j$ and $\Pi'|_i = \Pi|_j$, $\Pi'|_j = \Pi|_i$. Assume without loss of generality that $\Pi|_i = a_i$ and $\Pi|_j = a_j$. Denote by $t_l^{a_i a_j}$ the minimal number of votes for a_j needed in J_l such that the winner in J_l is a_j when the only candidates represented in the individual vote profile $\pi|_{J_l}$ are a_i and a_j . By MinRep, $n_l \geq t_l^{a_i a_j} \geq 1$. The federal profile Π can be obtained with an individual vote profile π such that $\pi|_{J_k} = \underbrace{\Pi|_k \dots \Pi|_k}_{n_k}$ for all $k \neq i, j$ and

$$\pi|_{J_i} = \underbrace{a_i \dots a_i}_{n_i - t_i^{a_i a_j} + 1} \underbrace{a_j \dots a_j}_{t_i^{a_i a_j} - 1}, \quad \pi|_{J_j} = \underbrace{a_j \dots a_j}_{n_j - t_j^{a_j a_i} + 1} \underbrace{a_i \dots a_i}_{t_j^{a_j a_i} - 1}.$$

Indeed, with this profile π , the winner in any J_k , $k \neq i, j$ is $\Pi|_k$ by MinRep, while the winner in J_i is a_i and the winner in J_j is a_j . Now, by swapping a vote for a_i in J_i with a vote for a_j in J_j , we get the federal profile Π' . By SwPr, it must be the case that $g(\Pi) = g(\Pi')$.

Q.E.D

Lemma 2 *Let A be a finite set of candidates. If C satisfies NoPiv, MinRep and SwPr then for any two federal profiles Π and $\Pi' \in A^m$ such that $X(\Pi) = X(\Pi')$, we have $g(\Pi) = g(\Pi')$.*

If $\text{Card}(X(\Pi)) = m$ then the result is straightforward using the previous lemma. Now consider that $\text{Card}(X(\Pi)) = s < m$. Then it must be the case that at least one candidate at the federal level is present more than once. We will show that if a local winner which is present more than once in a federal profile is replaced by another candidate in $X(\Pi)$ then the outcome of the election will be unchanged.

By noticing that any two federal profiles Π and Π' such that $X(\Pi) = X(\Pi')$ and $\text{Card}(X(\Pi)) = s < m$ can be obtained one from the other by a succession of such changes, the proof will be complete.

Assume candidate a_j is present more than once in the federal profile Π and that $g(\Pi) = z$. We show that if one a_j in Π is changed for an a_i , with $a_i \in X(\Pi)$, then the winner will not change. By Federal Anonymity, if the profile Π elects z , any permutation of this profile over the jurisdictions also elects z . There are two cases.

Case 1: If there is a jurisdiction J_l and a number k such that $f_l(\underbrace{a_i \dots a_i}_k a_j \dots a_j) = f_l(\underbrace{a_i \dots a_i}_{k+1} a_j \dots a_j) = a_i$ (resp. such that $f_l(\underbrace{a_i \dots a_i}_k a_j \dots a_j) = f_l(\underbrace{a_i \dots a_i}_{k+1} a_j \dots a_j) = a_j$). Then consider the partition σ and the individual profile π such that $\pi|_{J_l} = \underbrace{a_i \dots a_i}_{k+1} a_j \dots a_j$ and $\pi|_{J_m} = \underbrace{a_i \dots a_i}_{t_m(\sigma, a_j, a_i)-1} a_j \dots a_j$. We get $f_l(\pi|_{J_l}) = a_i$ and $f_m(\pi|_{J_m}) = a_j$ (resp. a_j and a_j). Now by swapping an a_j in J_m for an a_i in J_l we get $\pi|_{J_l} = \underbrace{a_i \dots a_i}_k a_j \dots a_j$ and $\pi|_{J_m} = \underbrace{a_i \dots a_i}_{t_m(\sigma, a_j, a_i)} a_j \dots a_j$ which yields $f_l(\pi|_{J_l}) = a_i$ and $f_m(\pi|_{J_m}) = a_i$ (resp. a_j and a_i). Hence at the federal level one a_j has been replaced by an a_i and by SwPr the federal winner is the same.

Case 3: this is the complementary case, i.e. in every J_l we have that for any k , $f_l(\underbrace{a_i \dots a_i}_k a_j \dots a_j) \neq f_l(\underbrace{a_i \dots a_i}_{k+1} a_j \dots a_j)$. By MinRep, this implies that in every J_l , whenever the individual profile contains only a_i and a_j , then the winner is the candidate receiving an odd number of votes and n_l is odd. To be completed

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