The Borda Majority Count

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Abstract

In Arrow's framework for social choice the voters are supposed to give a *preference ordering* over the alternatives and hence they are not able to express indifferences between two or more voters. In the framework of Michel Balinski and Rida Laraki, called Majority Judgement, as well as in the framework of Warren D. Smith, called Range Voting, voters are supposed to give an *evaluation* of the candidates in some common language or grading system. Consequently, they can convey much more information than in the framework of Arrow. While Warren D. Smith takes for each candidate its average as final value, Balinski and Laraki take the median value for each candidate, in order to reduce the danger of manipulation. However, this brings along a number of counter-intuitive results. As an alternative, we propose in this paper to use a version of the Borda Count, but now in the framework of Balinski and Laraki. We show that the resulting *Borda Majority Count* avoids the counter-intuitive results and has a number of other nice properties as well.

1 Introduction

In the traditional frame work of Arrow [1], where voters are supposed to give a preference ordering over the alternatives, there is no social welfare function that satisfies all of Arrow's properties: Pareto Optimality, Independence of Irrelevant Alternatives (IIA) and non-Dictatorship. In Arrow's framework, the Borda count [9, 12, 13] avoids the most serious drawbacks of plurality voting, e.g., that the Condorcet loser may win. However, it is highly manipulable and not IIA, i.e., the outcome may depend on an irrelevant candidate (see also [8]).

In Arrow's framework for social choice theory, every voter is supposed to give a *ranking* of the alternatives in his own *private* language. For instance, if two voters express the same thing, say that they prefer A to B, they may mean quite different things: one may mean that he has a slight preference for A to B, while the other may mean that he finds A excellent and rejects B. In the framework of Balinski and Laraki [2, 3], all voters are supposed to give an *evaluation* of each alternative in a *common* language or grading system, understood by everyone in the society. In particular, in the latter, but not in the former framework, it is quite possible to give the same evaluation of two or more candidates. Unfortunately, although Balinski and Laraki's Majority Judgement (MJ) has nice properties, among others it is independent of irrelevant alternatives (IIA), it produces in particular cases some counter-intuitive results. For that reason we adapt the Majority Judgement as follows: we keep the common language or grading system, for instance {excellent, very good, good, acceptable, poor, reject}, but apply the Borda Count, where the grade 'excellent' is good for 5 points, 'very good' for 4 points, ..., and 'reject' for 0 points. So, in a sense, we apply Range Voting (RV), introduced by Warren D. Smith [10], but restrict the range of {99, ..., 0} to {5, 4, 3, 2, 1, 0}. As we shall explain further on, this restricts the possibilities for manipulation.

This paper is organized as follows. In Section 2 we explain shortly how the Majority Judgement theory works and discuss two tie breaking rules. In Section 3 we present six counter-intuitive results when Majority Judgement is applied. In the forth section we propose to apply the Borda count in the framework of Balinski and Laraki and call this aggregation method the *Borda Majority Count* (BMC). We show that this way of aggregating the evaluations of the voters avoids the counter-intuitive results mentioned and in addition has a lot of nice properties. In section 5 we summarize our results.

2 Majority Judgement and Range Voting

The traditional framework of social choice theory, based on rankings of the alternatives by voters, is riddled with impossibility theorems, saying roughly that a social ranking function or choice function with only nice properties cannot exist [1]. Balinski and Laraki [2] on the one hand and Warren D. Smith [10] on the other hand proposed a new framework in which voters are not asked to give their preferences over the alternatives, but instead they are asked to give evaluations of all candidates in a common language or grading system understood by everyone in society. This is what happens in many contests in real life. Notice that in this way the information provided by the voters is much more informative than in the traditional framework of Arrow. For instance, it enables the voters to express that they give the same evaluation to two or more candidates, something impossible in the traditional framework. Also two voters who both prefer A to B may express their opinion in more detail: one may judge that A is excellent and B is very good, while the other may judge that A is very good and B is very poor. The difference between Balinski and Laraki's *Majority Judgement* and Smith' *Range Voting* is that in the first case the median value, and in the second case the average, of the grades given to a candidate is taken as the final grade of that alternative.

Below we present Majority Judgement and Range Voting in a compact form. In the first case the occurrence of ties is quite likely and hence tie breaking rules are needed. Balinski and Laraki present two tie breaking rules, which, however, as we shall point out, may yield different results.

2.1 Majority Judgement

As in the traditional framework, we assume a finite set C of m competitors or alternatives, $a_1, a_2, \ldots, a_m$, and a finite set J of n judges, $1, 2, \ldots, n$. Furthermore, a *common language* L is a finite set of strictly ordered grades r_n, or an interval of the real numbers. We take $r_i \geq r_j$ to mean that r_i is a higher grade than r_j or $r_i = r_j$.

The input for a *social grading function* (SGF) is then an m by n matrix, called a *profile*, filled with grades r_{ij} in L, where r_{ij} denotes the grade that judge j assigns to alternative a_i. So, row i in the profile contains the grades given by the different judges to alternative a_i, while column j contains the grades that judge j gives to the different alternatives. A *social grading function* is a function F that assigns to every profile the final grade of every competitor.

More precisely, let $f : L^n \to L$, then the *social grading function* (determined by f) is the function $F : L^{m\times n} \to L^m$ such that

$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix} \to (f(r_{11}, r_{12}, \ldots, r_{1n}), \ldots, f(r_{m1}, r_{m2}, \ldots r_{mn}))$$

where $f(r_{i1}, r_{i2}, \ldots, r_{in})$ is called the final grade of competitor a_i.

Balinski and Laraki [2] take for f an order function. The *kth* order function f^k takes as input an n-tuple of grades and gives as output the *kth* highest grade. They show that these order functions are demonstrably best for aggregating. They argue that one should take the middlemost aggregation functions and call the resulting system *Majority Judgement.* Suppose $r_1 \geq r_2 \geq \ldots \geq r_n$. A *middlemost aggregation function* f is defined by $f(r_1, r_2, \ldots, r_n) = r_{(n+1)/2}$ when n is odd, and $r_{n/2} \geq f(r_1, r_2, \ldots, r_n) \geq r_{(n+2)/2}$ when n is even. So, when n is odd, the order function $f^{(n+1)/2}$ is the middlemost aggregation function. When n is even, there are infinitely many. In particular, the upper middlemost $f^{n/2}$, defined by $f^{n/2}(r_1, r_2, \ldots, r_n) = r_{n/2}$, and the lower middlemost aggregation function $f^{(n+2)/2}$, defined by $f^{(n+2)/2}(r_1, r_2, \ldots, r_n) = r_{(n+2)/2}$. Any grade not bounded by the middlemost aggregation functions is condemned by an absolute majority of judges as either too high or too low. The *majority grade* $f^{maj}(a_i)$ of candidate a_i is by definition equal to $f(r_{i1}, r_{i2}, \ldots, r_{in})$, where f is the lower middlemost aggregation function and $r_{i1}, r_{i2}, \ldots, r_{in}$ are the grades given to a_i by the voters or judges.

When the number of voters is odd, the majority grade is the median, or the one middle grade. In the case of an even number of voters and a candidate's two middle grades are different, Balinski and Laraki argue that the lower of the two middle grades must be the majority grade.

The majority grade of a candidate is his or her median grade. It is simultaneously the highest grade approved by a majority and the lowest grade approved by a majority. For instance, if a candidate has got the grades, 10, 8, 7, 5, 2 on a scale of 1 till 10, his majority grade will be 7, since there is a majority of judges who think the candidate should have at least grade 7, and an other majority of judges who think the candidate should have at most grade 7.

Let p be the number of grades given to a candidate above its majority grade α, and q be the number of grades given to the same candidate below its majority grade α. Then

$$\alpha^* = \begin{cases} \alpha^+ & \text{if} \quad p > q; \\ \alpha^\circ & \text{if} \quad p = q; \\ \alpha^- & \text{if} \quad p < q \end{cases}$$

2.2 Tie-breaking rules

In [2] the majority grades of the candidates are used to calculate the majority ranking. The general *majority ranking* $>_{maj}$ between two competitors A and B is determined as follows:

- if $f^{maj}(A) > f^{maj}(B)$, then $A >^{maj} B$.
- If $f^{maj}(A) = f^{maj}(B)$, then drop one majority grade from the grades of each competitor and repeat the procedure.

In [3] Balinski and Laraki present another tie braking rule for the case of large elections. Three values attached to a candidate, called the candidate's *majority value*, are sufficient to determine the candidate's place in the majority ranking:

$$(p, \alpha, q) \;\; \text{where} \begin{cases} p = \text{number of grades above the majority grade} \\ \alpha = \text{majority grade, and} \\ q = \text{number of grades below the majority grade} \end{cases}$$

In [3] the order between two majority values is now defined as follows:

$$(p, \alpha^*, q) > (s, \beta^*, t) \text{ if } \alpha^* > \beta^*$$

where $\alpha^* > \beta^*$ if $\alpha > \beta$ and $\alpha^+ > \alpha^\circ > \alpha^-$.

Now suppose α^* and β^* are the same. Then

$$(p, \alpha^+, q) > (s, \alpha^+, t) \text{ if } \begin{cases} p > s, \text{or} \\ p = s \text{ and } q < t \end{cases}$$

$$(p, \alpha^-, q) > (s, \alpha^-, t) \text{ if } \begin{cases} q < t, \text{or} \\ q = t \text{ and } p > s \end{cases}$$

$$(p, \alpha^\circ, q) > (s, \alpha^\circ, t) \text{ if } p < s, \text{ where } p = q \text{ and } s = t.$$

Note that the tie braking rules in [2] and [3] are different. Consider a simple example:

	p	Excellent	V.Good	Good	Accepted	Poor	Rej	q	Total
A	37	17	20	**24**	0	16	23	39	100
B	36	16	20	**25**	21	17	1	39	100

A and B have both majority grade* Good$^-$. According to the simplified tie breaking rule in [3], the majority value of A is (37, Good$^-$, 39) and for B is (36, Good$^-$, 39). Now, for A the number of grades above the majority grade (37) is higher than for B (36). Hence, A is the winner. However, if we apply the general tie breaking rule in [2], by dropping the majority grade one by one, then B turns out to be the winner.

Consider, for instance, a similar example for a large election:

	Excellent	V.Good	Good	Accepted	Poor	Rejected	Total
A	18758	25242	**35984**	10016	9126	874	100000
B	29818	14182	**34952**	11048	9478	522	100000

A's majority value is (44000, Good, 20016) and his majority grade* is Good$^+$. B's majority value is (44000, Good, 21048) and his majority grade* is also Good$^+$. A and B have the same grades, so, by the tie breaking rule in [3], the winner is A, because $20016 < 21048$. However, when we apply the general tie breaking rule in [2], we find B as the winner. These examples clearly show that the tie breaking rules in [2] and [3] may give different results.

2.3 Range Voting

Warren D. Smith takes for the language L the set of grades from 0 till 99 and for $f(r_{i1}, r_{i2}, \ldots, r_{in})$ the average of the grades $r_{i1}, r_{i2}, \ldots, r_{in}$ given to the alternative a_i. The resulting system is called *Range Voting*. In [10] he gives an exposition of his Range Voting and a comparison with many other election mechanisms. It must be admitted that Range Voting has many properties not shared by other election mechanisms. But, of course, Range Voting is vulnerable for manipulation: voters who have a slight preference for A over B might strategically give 1 point to B and 99 to A in order to achieve that their favored candidate wins. That is precisely why Balinski and Laraki take the median value of the grades given to a candidate as the final grade of that candidate. In this way they make Majority Judgement (much) less vulnerable for manipulation. An advantage of Range Voting is that the probability of ties is small or very small, in particular when the number of voters or judges is large. A simple solution in the unexpected case of a tie is then simply tossing a coin.

A special case of Range Voting is Approval Voting [5], where the range consists of 0 (disapproval) and 1 (approval).

3 Paradoxes in Majority Judgment

The voting paradoxes play an important role in social choice theory and more generally in philosophy. They are an important source of progress in science. Already the founding fathers of social choice theory, Marquis de Condorcet (a French mathematician, philosopher, economist, and social scientist), and Jean-Charles de Borda (a French mathematician, physicist, political scientist, and sailor), have drawn attention to some paradoxes, i.e. counterintuitive results. Also Arrow's result of the 1950's was considered to be paradoxical, because at first sight the three conditions of Pareto Optimality, Independence of Irrelevant Alternatives and Non-Dictatorship do not seem to be inconsistent. But Arrow showed that under some mild conditions they are (together) inconsistent.

Unfortunately, it turns out that also Majority Judgement may yield some counter-intuitive results. Below we point out a number of such examples.

Paradox 1 *Majority winner may loose*

Suppose there are two alternatives A and B and one hundred judges, whose evaluations are summarized in the following table.

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	49	19	30	**51**	0	0	0	0
B	50	19	31	**20**	15	8	7	30

According to Balinski's majority judgment theory [2, 3], the (lower) middle grade of both A and B is Good, A's majority value is (49, Good$^+$, 0) and B's majority value is (50, Good$^+$, 30). According to both tie breaking rules B is the winner, although there is no judge who gives A a grade lower than Good, while thirty judges do so for B. That B is the winner is the result of the one extra voter that judges B as Very Good and not taking into account the 30 voters who evaluate B lower than Good. Considering the cumulative majority judgement grades shown in the table below, one would expect A to be the winner.

At Least	Excellent	V.Good	Good	Accepted	Poor	Rejected
A	19	49	100	100	100	100
B	19	50	70	85	93	100

Paradox 2 *Looser is the Majority Judgement winner*

Majority judgement theory may produce counterintuitive results for small choices, like in figure skating, gymnastic competition, piano competition, wine competition etc. Consider, for instance, a shooting contest with four players A, B, C and D and one judge; they played ten rounds and the following grades were obtained.

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	5	2	2	1	0	0	**5**	0
B	5	0	2	2	1	0	**5**	0
C	5	0	1	2	1	1	**5**	0
E	0	0	0	0	0	**6**	4	4

Intuitively, one would expect A to be the winner. But because Balinski and Laraki take the lower middle grade as the majority grade, the majority grade of A, B and C is Rejected. However, because the majority grade of E is Poor, E becomes the majority judgement winner. But A performs five times Good or better, while E's best performance is Poor.

Paradox 3 *Adding two voters both rejecting all candidates may change the outcome*

Consider the following example with two candidates A and B and ten judges:

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	3	1	2	**5**	1	1	0	2
B	2	0	2	**7**	1	0	0	1

A and B have both majority grade Good and their majority values are respectively (3, Good$^+$, 2) and (2, Good$^+$, 1). According to both tie breaking rules, A is the Majority Judgement winner. Suppose now we add two ballots with Rejected for both A and B:

	p	Excellent	V.Good	Good	Accepted	Poor	Rejected	q
A	3	1	2	**5**	1	1	2	4
B	2	0	2	**7**	1	0	2	3

Then under both the original tie breaking rule of Majority Judgement in [2] and the simplified tie breaking rule of [3] now alternative B will win.

Another example is due to Dan Bishop [4] and Rob Lanphier [7]: suppose two candidates A and B, grades from 0 to 6 and the following election results:

A's scores are 1, 2, 4, 4, 6 Majority grade of A is 4.
B's scores are 2, 3, 3, 6, 6 Majority grade of B is 3.

So, A is the majority judgement winner. However, if a zero ballot for both A and B is added, then B becomes the winner:

A's scores become 0, 1, 2, 4, 4, 6 Majority grade of A is 2.
B's scores become 0, 2, 3, 3, 6, 6 Majority grade of B is 3.

That A is no longer the winner is caused here by the fact that the majority grade is by definition the lower middle grade in case the number of voters is even.

Paradox 4 *No show paradox*

Suppose two friends Romeo and Julia decided not to cast their votes because Romeo is in favor of A and wants to give A one grade higher than B, say 6 and 5 respectively, and Julia is in favor of B and wants to give B one grade higher than A, 6 and 5 respectively. Since they expect that their votes will cancel each other out, they decided not to cast their votes. However, let us see what happens if we add their votes to the example of Bishop and Lanphier above. Then B becomes the winner instead of A:

A's scores become 1, 2, 4, 4, 5, 6, 6 Majority grade of A is 4.
B's scores become 2, 3, 3, 5, 6, 6, 6 Majority grade of B is 5.

So, B instead of A becomes the majority judgement winner.

Paradox 5 *Increased support for a candidate may turn him from a winner into a looser.*

Starting with the example of Bishop and Lanphier again, suppose that Romeo and Julia are two new voters who both give to the winner A grade 6 and to B grade 5 only. Then we have the following situation:

A's scores become 1, 2, 4, 4, 6, 6, 6	Majority grade of A is 4.
B's scores become 2, 3, 3, 5, 5, 6, 6	Majority grade of B is 5.

Now again B becomes the winner instead of A.

Here is another example. Suppose a public service committee of 10 members has interviewed two candidates A and B for the post of an official. Nine members gave their evaluation as follows:

	p	Exc	V.Good	Good	Accept	Poor	Reject	q
A	3	1	2	**3**	3	0	0	3
B	4	2	2	**1**	2	2	0	4

Applying Majority Judgement with the original tie breaking rule from [2], candidate A will be the winner. The nine members report this result to the chairman of the committee for a final decision. The chairman is also in favor of candidate A, and gives grade Excellent to A and grade Very Good to candidate B. With this increased support for A one expects that candidate A shall win. However, let us see what happens:

	p	Exc	V.Good	Good	Accept	Poor	Reject	q
A	4	2	2	**3**	3	0	0	3
B	5	2	3	**1**	2	2	0	4

Now, applying Majority Judgement with the original tie breaking rule from [2], we find to our surprise that B instead of A wins.

Before presenting the next paradox we define the notions of winner consistent and rank consistent.

Winner consistent: If there are two separate parts of an electorate and a candidate wins in both electorates, then he must win in the whole electorate as well.

Rank consistent: If there are two separate parts of a constituency and the rankings of two candidates a, b in both parts of the constituency are $a > b$, then in the whole electorate the ranking of the candidates will be the same, i.e., $a > b$ as well.

Paradox 6 *Two districts paradox: Majority Judgement is not winner and not rank consistent*

Consider the following example with two candidates A and B, language {1, 2, 3, 4, 5, 6}, and two districts.

In District-I:

A's scores are	6, 4, 3, 3, 1	Majority grade of A is 3.
B's scores are	6, 6, 2, 2, 1	Majority grade of B is 2.

In District-II:

A's scores are	6, 6, 5, 2, 2	Majority grade of A is 5.
B's scores are	6, 6, 4, 4, 1	Majority grade of B is 4.

So, A wins in each of both districts. But if we combine the two districts, we get:

A's scores are	6, 6, 6, 5, 4, 3, 3, 2, 2, 1	Majority grade of A is 3.
B's scores are	6, 6, 6, 6, 4, 4, 2, 2, 1, 1	Majority grade of B is 4.

So, A wins in each district I and II, while B wins when the results of both districts are joined together. In both districts we have $a >^{maj} b$, but in the combination of the two districts we have $b >^{maj} a$. In other words, Majority Judgement is neither winner nor rank consistent. By the way, Balinski and Laraki claim in [2] that one can not expect that Majority Judgement is winner consistent.

Here is another example. Suppose there are two parts of an electorate, part I and part II. In part I, there are two candidates and one thousand (1000) judges, giving their judgements as follows:

Part I		p	Exc	V.Good	Good	Accept	Poor	Reject	q
	A	245	245	**256**	89	119	128	163	499
	B	400	156	244	**301**	199	79	21	299

The majority-grade of candidate A is Very Good and that of B is Good. Also, the majority value of A, (245, V.Good$^-$, 499), is higher than the majority value of B, (400, Good$^+$, 299). So, $A >^{maj} B$. In part II, there are the same two candidates and fifteen hundred (1500) judges, giving their judgements as follows:

Part II		p	Exc	V.Good	Good	Accept	Poor	Reject	q
	A	400	78	125	197	**389**	588	123	711
	B	699	145	325	125	104	**275**	526	526

The majority-grade of candidate A is Accept and that of B is Poor. Note that also the majority value of A, (400, Accept$^-$, 711), is higher than that of B, (699, Poor$^+$, 526). So, also in Part II, $A >^{maj} B$. Now let us look at the result in the combined electorate, as shown below:

Part I + Part II

	p	Exc	V.Good	Good	Accept	Poor	Reject	q
A	990	323	381	286	**508**	716	286	1002
B	870	301	569	**426**	303	354	547	1204

In the whole electorate, the final majority-grade of candidate A is Accept and that of B is Good; Also, the majority value of A, (990, Accept$^-$, 1002), is now strictly lower than that of B, (870, Good$^-$, 1204). So, while $A >^{maj} B$ in both parts I and II, we have $B >^{maj} A$ in the whole electorate.

Grade consistent: If there are two separate parts of an electorate and the majority grade of a candidate in each is α, then the majority grade of the candidate is α in the whole electorate as well.

Theorem 1 *Majority Judgment theory is grade consistent.*

Proof: Suppose the majority grade of a candidate A is α in both districts I and II, that is, in district I there is a majority M_1 that judges A deserves at least grade α and another majority M_2 that judges A deserves at most grade α. Similarly, in district II there are majorities N_1 and N_2. Then in the union of I and II, $M_1 \cup N_1$ is a majority that judges that A deserves at least grade α, and $M_2 \cup N_2$ is a majority that judges that A deserves at most grade α. Thus α is the majority grade of A in the whole electorate as well. □

4 The Borda Majority Count and its Properties

In 1770 the French mathematician Jean-Charles de Borda proposed the Borda count as a method for electing members of the French Academy of Sciences. In the framework of Arrow, voters are supposed to give a *preference ordering* over the alternatives, everyone in his or her own *private language.* If a voter ranks the alternatives $a_1, \ldots, a_m$ as $a_{\sigma(1)} > a_{\sigma(2)} > \ldots > a_{\sigma(m)}$ (where σ is a permutation of $\{1, \ldots, m\}$), $a_{\sigma(1)}$ gets $m-1$ Borda points, $a_{\sigma(2)}$ gets $m-2$ Borda points, $\ldots$, $a_{\sigma(m)}$ gets 0 Borda points. The *Borda score* of a given alternative a is the total number of Borda points given by the voters to a. The social ranking of the alternatives and the winner(s) are obtained by comparing the Borda scores of the different alternatives.

As explained in Section 2, Michel Balinski and Rida Laraki [2, 3] use a different framework, in which voters are supposed to give an *evaluation* of each candidate in some *common language* or grading system. Hence, the voters are enabled to provide much more information than in Arrow's framework. As explained in section 3, unfortunately this method produces some counter-intuitive results (see also [14, 6]), partly due to its median based aggregation method. For that reason, we propose to use Borda's aggregation method, but now applied in the framework of Balinski and Laraki [2].

4.1 The Borda Majority Count (BMC)

Definition 1 *Let a be an alternative and let $\{g_1, g_2, ..., g_k\}$ be the set of grades, with $g_1 < g_2 < \ldots < g_k$. Let p_i be the number of voters who gave grade g_i to a, where $i = 1, 2, \ldots, k$. Then we define $BMC(a) := p_1 \cdot 0 + p_2 \cdot 1 + \ldots + p_k \cdot (k-1)$ and call it the Borda Majority Count (BMC) of a.*

$$BMC(a) = \sum_{i=1}^{k} (i-1) \cdot p_i$$

For instance, suppose we have six grades: Reject, Poor, Acceptable, Good, Very Good, and Excellent. Then we assign 0 points to the Reject grade, 1 point to Poor, 2 points to Acceptable, 3 points to Good, 4 points to Very Good, and 5 points to Excellent, respectively.

Given a finite set of m candidates $C = \{c_1, c_2, \ldots, c_m\}$, a finite set of n voters $V = \{v_1, v_2, \ldots, v_n\}$ and a common language L, i.e., a set of grades, for instance {Reject, Poor, Acceptable, Good, Very Good, Excellent}, an input *profile* is an m by n matrix of grades where each row i contains the grades given by the voters to candidate c_i and each column j contains the grades voter v_j assigns to the candidates. Balinski and Laraki [2, 3] define a *social grading function* F as a function $F : L^{m \times n} \to L^m$ which assigns to any $m \times n$ matrix α the output

$$F(\alpha) = (f^{maj}(c_1), f^{maj}(c_2), \ldots, f^{maj}(c_m))$$

where $f^{maj}(c_i)$ is the majority grade, i.e., the median value, of candidate c_i.

The *Borda Majority Count (BMC)*, on the other hand, is a function $BMC : L^{m \times n} \to \mathbb{N}^m$ that assigns to any profile α the output $(BMC(c_1), \ldots, BMC(c_m))$, where $BMC(c_i)$ is the Borda Majority Count of candidate c_i.

Example 1 Next we apply the Borda Majority Count to the experimental results in the French presidential election 2007. Balinski and Laraki [3] obtained the following data in the three precincts of Orsay:

	Exc	V.Good	Good	Accept	Poor	Reject	BMC
Besancenot	4.1	9.9	16.3	16.0	22.6	31.1	163.3
Buffet	2.5	7.6	12.5	20.6	26.4	30.4	148.0
Schivardi	0.5	1.0	3.9	9.5	24.9	60.4	62.1
Bayrou	13.6	30.7	25.1	14.8	8.4	7.4	**304.1**
Bové	1.5	6.0	11.4	16.0	25.7	39.5	123.4
Voynet	2.9	9.3	17.5	23.7	26.1	20.5	102.1
Villiers	2.4	6.4	8.7	11.3	15.8	55.5	102.1
Royal	16.7	22.7	19.1	16.8	12.2	12.6	**277.4**
Nihous	0.3	1.8	5.3	11.0	26.7	55.0	73.3
Le Pen	3.0	4.6	6.2	6.5	5.4	74.4	70.4
Laguiller	2.1	5.3	10.2	16.6	25.9	40.1	121.4
Sarkozy	19.1	19.8	14.3	11.5	7.1	28.2	**249.2**

Using the Borda Majority Count, we get approximately the same rank order as by using Majority Judgement, except for Voynet whose rank according to the Borda Majority Count is greater than according to the Majority Judgement.

The Borda Majority Count aggregation resembles Range Voting [10], the main difference being that it uses a different range. In the Borda Majority Count, we have restricted the range from zero to five, while Warren D. Smith uses a range from 0 to 99 in Range Voting [10]. In the Borda Majority Count, the evaluation of the alternatives by the individuals is in terms of a common language of grading, for instance {Reject, Poor, Acceptable, Good, Very Good, Excellent}, while in Range Voting it is expressed by numbers instead of words. So, in fact we combine ideas from Majority Judgement [2, 3] and from Range Voting [10], by taking the language from Balinski and Laraki [2, 3] and by adding the numbers obtained (5 for Excellent, 4 for Very Good, etc) as is done in Range Voting [10], instead of taking the median value as is done by Balinski and Laraki [2, 3].

4.2 The BMC Tie Braking rule

When we calculate the Borda Majority Count, it is possible that two or more alternatives have the same Borda Majority Count, in other words that there is a tie. There is a simple way to brake the tie, by dropping the Reject grades and re-calculating the Borda Majority Count. If there is still a tie, then drop the Poor grades and re-calculate the Borda Majority Count, and we continue this process by dropping grades step by step from lower to higher until the tie is broken.

The candidate who has the greatest Borda Majority Count, is the *Borda Majority winner*. In general, the *Borda Majority* ranking $>_{BMC}$ between two competitors is determined as follows:

- If $BMC(c_i) > BMC(c_j)$, then $c_i >_{BMC} c_j$.
- If $BMC(c_i) = BMC(c_j)$, then drop all the reject grades and recalculate the Borda Majority Count. The procedure is repeated step by step by dropping grades from lower to higher until a winner among c_i and c_j is found.

Example 2 Consider the case of one hundred judges who gave their judgements for three candidates, as follows.

	Exc	V.G	Good	Acc	Poor	Rej	BMC-I	BMC-II	BMC-III
a	11	33	21	29	2	4	310	214	120
b	13	34	18	24	7	4	310	214	**125**
c	13	29	22	31	1	4	310	214	119

In this example, we see that the alternatives a, b and c have the same Borda Majority Count BMC-I. There is a tie, so, we drop the Reject grades of all alternatives and recalculate the Borda Majority Count; once again all alternatives have the same Borda Majority Count BMC-II. Now dropping the Poor grades and recalculating the Borda Majority Count, we see that alternative b has the greater Borda Majority Count BMC-III and wins the election.

4.3 Properties

Because the Borda Majority Count is a special case of Range Voting, it does have all the nice properties of Range Voting as explained in [11]. In particular, it does have the following properties.

Participation property: Casting an honest vote can never cause the election result to get worse (in the voter's view) than if she hadn't voted at all. So, the

no-show paradox, where some class of voters would have been better off by not showing up, does not occur.
Favorite-safe: It is never more strategic to vote a non-favorite ahead of your favorite. In other words, it is safe to vote for your favorite.
Clone-safe: If a 'clone' of a candidate (rated almost identical to the original by every voter) enters or leaves the race, that should not affect the winner (aside from possible replacement by a clone).
Monotonicity: If somebody increases their vote for candidate c (leaving the rest of their vote unchanged) that should not worsen c's chances of winning the election, and if somebody decreases their vote for candidate b (leaving the rest of their vote unchanged) that should not improve b's chances of winning the election.
Remove-loser safe: If some losing candidate X is found to be a criminal and ineligible to run, then the same ballot is still useable to conduct an election with X removed, and should still elect the same winner.
Precinct-countable: If each precinct can publish a succinct summary of the vote (sub)total in that precinct, then the overall country-wide winner can be determined from those precinct subtotals.

In addition, the Borda Majority Count satisfies the following properties.
Neutrality: All candidates are treated equally.
Independent of Irrelevant Alternatives (IIA): The rank order of two alternatives is not influenced by a third one.
Pareto condition: If every voter assigns higher or equal grades to candidate c_1 than to candidate c_2, then c_1 will also be collectively preferred to c_2, i.e,. $c_1 \geq_{BMC} c_2$.
Anonymity: The Borda Majority Count prevents unequal treatment of voters. It erects a barrier to any form of discrimination and treats all voters the same.
Unanimity: The Borda Majority Count is unanimous: if every voter gives to candidate c a higher grade than to all other candidates, then c wins.
Transitive: The Borda Majority Count is transitive: if candidate c_1 is weakly preferred to candidate c_2, i.e., $c_1 \geq_{BMC} c_2$ and c_2 is weakly preferred to c_3, i.e., $c_2 \geq_{BMC} c_3$, then c_1 is weakly preferred to c_3, i.e., $c_1 \geq_{BMC} c_3$.
No-Dictator: The Borda Majority Count has the property to avoid dictatorship. One can't overcome the societal preference.

Theorem 2 *The Borda majority count is winner and rank consistent.*

Proof: Suppose there are two parts of a constituency, say part-I and part-II, and $a >^{I}_{BMC} b >^{I}_{BMC} c$ and in part-II, $a >^{II}_{BMC} b >^{II}_{BMC} c$. Then by definition, $BMC_I(a) > BMC_I(b) > BMC_I(c)$ and $BMC_{II}(a) > BMC_{II}(b) > BMC_{II}(c)$. Then clearly $BMC_{I \cup II}(a) > BMC_{I \cup II}(b) > BMC_{I \cup II}(c)$, i.e., $a >^{I \cup II}_{BMC} b >^{I \cup II}_{BMC} c$. In particular, if a wins in both parts I and II, then a wins in the union of part-I and part-II. Thus the Borda Majority Count is winner and rank consistent. □

A weak point of Range Voting and of the traditional Borda count as well is its sensitivity for manipulation. For instance, a voter who favors candidate A and knows candidate B is a serious competitor of A, may give to B say 1 point instead of the 90 points if he were honest, in order to achieve that his most favorite candidate A will win. By restricting the range to {0, 1, 2, 3, 4, 5} instead of {0, ..., 99} the effects of manipulation in the Borda Majority Count are less dramatic.

Proposition 1 *The Borda Majority Count is less vulnerable to manipulation than Range Voting and the original Borda count.*

Proposition 2 *All paradoxical results mentioned in Section 3 disappear if the Borda Majority Count is applied instead of Balinski and Laraki's Majority Judgement.*

Proof: The reader can easily check for himself. □

Extremist/Centrist bias Warren D. Smith describes this in [11] as follows: 'Suppose the candidates are positioned along a line (1-dimensional) or in a plane (2-dimensional) and voters prefer candidates located nearer to them. In some voting systems, it is usually difficult or impossible for 'centrists' (centrally located relative to the other candidates) to win. Those systems 'favor extremists'. Other voting systems 'favor centrists'. We apologize for defining this property rather vaguely, but in practice it is often quite clear which category a voting system belongs in.' He argues that the performance of the Borda Majority Count with respect to this Extremist/Centrist bias is okay. Note that the Borda Majority Count winner is not necessarily the Condorcet winner. For instance, consider the case of two candidates A and B and three voters, two of which give to A grade Excellent and to B grade Very Good, and the third voter gives to A grade Reject and to B grade Good. Then A is the Condorcet winner, while B is the Borda Majority Count winner. The reason behind this is that face to face confrontations ignore how the electorate *evaluates* the respective candidates except, of course, that one is evaluated higher than the other.

Up till now we have used the Borda Majority Count for the selection of a winner. But it is also straightforward how to use it for the selection of a parliament or a House of Representatives. We can calculate the number of seats for each party by the following simple rule, for the moment ignoring rounding off problems. Let C be the set of candidates and $a \in C$.

$$\text{Allocation of seats to party } a = \frac{BMC(a) \cdot \text{total number of seats}}{\sum_{c \in C} BMC(c)}$$

Example 3 Suppose there are seven parties, A, B, C, D, E, F, G, one hundred and fifty (150) seats in the House of Representatives and one hundred voters, who evaluate the different parties as shown below. Then the allocation of seats according to the formula above is as shown in the last column below.

	Exc	V.Good	Good	Accept	Poor	Reject	BMC	Seats
A	12	15	26	16	10	21	161	24
B	10	17	30	21	12	10	172	26
C	13	18	28	25	16	0	187	28
D	2	14	43	35	4	2	171	26
E	4	15	28	42	10	1	159	24
F	3	5	10	15	25	42	62	9
G	1	8	12	35	36	8	87	13
Sum							999	150

5 Conclusion

The Borda count in the framework of Arrow requires of every voter that he gives a *ranking* of the candidates in his own *private* language. The voter has, for instance, no possibility to express an indifference between two or more candidates. In the framework of Balinski and Laraki on the one hand and of Warren D. Smith on the other hand, every voter is required to give an *evaluation* of each candidate in a *common* language or grading system, well understood by everyone in the society. In this way the voter is able to convey much more information than in the framework of Arrow. In Warren D. Smith' Range Voting the common language is the set $\{0, \ldots, 99\}$ and the final grade of a candidate is the average (or sum) of the grades assigned to it. Balinski and Laraki propose the set {Excellent, Very Good, Good, Acceptable, Poor, Reject} as the common language and take as the final grade of a candidate the median value of the grades assigned to that candidate.

Warren D. Smith' Range Voting has many nice properties, but may suffer strongly of the possibility of manipulation; for instance, giving a competitor 1 point instead of the 90 if one were honest. By taking the median value instead of the average, Balinski and Laraki's Majority Judgement is more robust with respect to manipulation, but the price to be paid is a number of counterintuitive results, as explained in Section 3. For that reason we propose to use the common language of Balinski and Laraki, to identify grade Excellent with the number 5, grade Very Good with 4, Good with 3, Accept with 2, Poor with 1 and Reject with 0, and to add the numbers assigned to a candidate c, calling the result the Borda Majority Count of c.

Since the Borda Majority Count is a special case of Range Voting, now with $\{0, 1, 2, 3, 4, 5\}$ as common language, it inherits all the nice properties of Range Voting, at the same time avoiding all the counterintuitive results mentioned in Section 3. In addition, by restricting the common language to only six grades instead of hundred, the possibilities for manipulation are seriously restricted. The Borda Majority Count is independent of irrelevant alternatives (IIA), rank and winner consistent. It is also easy to use it for the assignment of seats to parties in a parliament.

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