The Traveling Group Problem

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Abstract

This paper combines social choice theory with mathematical optimization by applying various group decision concepts to a classical problem of combinatorial optimization, namely the famous traveling salesperson (salesman) problem. The aim of the latter is to find a tour through all vertices of a given graph along edges of minimal total cost. In this contribution we replace the measure of additive edge costs by the social acceptance of different edges and the resulting tours. In particular, for four different voting rules, the Borda rule, approval voting, plurality rule and simple majority rule, we will investigate the social acceptance of tours derived from global and local decisions. It will be shown that these two decision approaches can lead to widely varying results.

1 Introduction

The areas of social choice and mathematical optimization are traditionally completely separated and pursued by nonintersecting communities with very few exceptions. While this fact is historically easy to understand it is also somewhat suprising since both fields are based on the quest for optimal decisions. In this contribution we try to apply a group decision process to a classical graph problem which is widely studied in combinatorial optimization. Our approach is firmly based on social choice theory and only takes a problem from optimization as an illustration of how collective decisions may work in a new context.

The main goal of this paper is to analyse a group decision process in a framework similar to the traveling salesperson problem (TSP). In the latter the aim is to find a tour of minimal cost. In this paper we focus on the social acceptance of different tours.

Consider a group that wants to jointly visit a number of cities. However, the members of the group might have different preferences over the routes traversed between one city and another. As for visiting n cities there exist $(n-1)!$ tours, finding the tour of maximal social acceptance seems to be a difficult task.[1] The group has essentially

[1]E.g. with 12 cities there would exist almost 40 million different tours but "only" 66 different routes between two cities.

two decision possibilities. Either they decide "globally" upon the shape of the tour, i.e. once before they start it, or "locally", i.e. sequentially whenever they have reached a new city. One purpose of this paper will be to indicate what differences in the shape of tours can arise in that respect using different voting rules to solve the group decision problem.

The general problem has certain resemblance to the bus touring problem [5] in which an optimal subset of routes between given tourist sites needs to be selected. However, our approach relies exclusively on ordinal and interpersonally non-comparable preferences. In a more professional sense one could think of a group of tourism managers who need to specify a sight-seeing tour based on certain locations that have to be visited. Their preferences, represented by ordinal rankings of the $\binom{n}{2}$ connections between two locations, might be different and therefore a certain kind of aggregation process seems compelling.[2]

Besides the analysis of traveling groups, tours can also be given a much more economics-like interpretation. Consider a production activity in which a good is produced in various steps, however, there is not necessarily a natural sequence in those production steps. A board of managers might have to decide on the production process given their invidiual (and possibly different) preferences over the changes from one production step to the other. As this transition from one step to the other might not be something objectively measureable (like money costs or time) but only subjectively measurable (like personal effort which might be different for different managers and different transitions from one step to the other), again an aggregation of individual preferences will be necessary.

In this paper we represent such situations by a complete graph where we assume that members of the group have preferences over the possible edges of a graph, given by ordinal rankings of those edges, and hence building the basis for the decision process. Decisions are supposed to be taken on a global or local basis, using one of four different voting rules used in the decision process. Those rules are the scoring rules like the Borda rule (alternatives are ranked according to their average positions in the individual rankings) and the Plurality rule (alternatives are ranked according to the number of top positions in the individual rankings), but also include Approval Voting (alternatives are ranked according to the number of individuals that approve of each alternative) and simple majority rule (alternatives are ranked according to pairwise majority comparisons). Using those voting rules in combination with greedy algorithms we investigate to what extent tours determined on a local basis (local tours) will be different from tours determined on a global basis (global tours). In addition, we compare local and global tours with respect to (individual pairwise) dominance.

That decisions taken on a global basis can be different from those taken on a local basis for many voting rules and in particular scoring rules, is something that is related to the work of Don Saari ([21], [22]). With his dictionaries of voting paradoxes he essentially showed that anything can happen. Especially the undesirability of situations in which group decisions change by deleting irrelevant alternatives from the set to be

[2]Another example in sports comes from team orienteering events (cf. Golden et al. [8])). A team only knows the fixed locations on a map that need to be passed jointly. As the members of a team might have different valuations of the degree of difficulty of the routes between those locations, this sets the necessity for a group decision.

voted upon is something that has been intensively discussed since Arrow's impossibility theorem [1] under the name of independence of irrelevant alternatives ([19], [23]).

In this paper, besides discussing such issues in a new framework, we want to go beyond the existing analysis by introducing also other voting rules like Approval voting and simple majority rule, the latter being immune to irrelevant alternatives. Moreover, not only will our focus be on showing extreme outcomes[3] but also on the individual evaluation of such outcomes. In particular, we will show that for the Borda rule and Approval voting, the global tour can be pairwise dominated by the local tour and vice versa. This is not the case for the Plurality rule where the global tour cannot be pairwise dominated by any other tour. Using simple majority rule, global and local tours can never be disjoint, however they can be pairwise dominated by other tours.

The paper is structured as follows. Section 2 provides the formal framework, i.e. the definitions of voting rules, tours and dominance concepts. Section 3 first introduces the local and the global greedy rule and then states and proves the results with respect to the Borda rule, Approval voting and other voting rules. The final section concludes the paper

2 Formal Framework

Let $I = \{1, 2, ..., m\}$ denote a finite set of individuals and let $G = (V, E)$ be a complete undirected graph where V denotes a finite set of k objects and E is the set of all two-element subsets of V. An element $e = \{v, w\} \in E$ with $v, w \in V$ is the edge between the objects v and w.[4] Basics of graph theory can be found in [10]. Preferences over the set of edges, E, are represented by complete, transitive and reflexive binary relations, $\succsim$, with asymmetric and symmetric parts $\succ$ and $\sim$, respectively. The set of all such preferences is given by $\mathcal{B}$. Individual preferences are assumed to be complete, transitive and asymmetric binary relations and will be written as $\succ_i$. Let $\mathcal{P}$ denote the set of admissible profiles, with generic entries $P = (\succ_1, \succ_2, ..., \succ_n) \in \mathcal{P}$.

For any pair $e_j, e_k \in E$ let $p_{j,k} = |\{i \in I : e_j \succ_i e_k\}| - |\{i \in I : e_k \succ_i e_j\}|$. Then e_j is socially at least as good as e_k if $p_{j,k} \geq 0$. A preference with less structure can be determined by partitioning the set E, for each $i \in I$, into a set $S_i \subset E$, i.e. edges that i approves of, and a set $E \backslash S_i$, i.e. edges that i disapproves of. We call S_i sincere[5] if for all $a \in S_i$ and $b \in E \backslash S_i$, $a \succ_i b$. Furthermore, let the singleton set $S_i^t = \{a \in E : \forall b \in E, a \succ_i b\}$ denote individual i's top alternative in the ranking $\succ_i$. Finally, preferences restricted to subsets of E, $T \subset E$, will be denoted by $\succ_i |T \equiv \succ_i \cap T^2$, restricted profiles will be denoted by $P|T = (\succ_1 |T, \succ_2 |T, ..., \succ_n |T)$.

[3]For comparisons of voting rules see Klamler [11] and Ratliff [18]. For a general discussion on voting paradoxes we recommend the work by Hannu Nurmi ([14],[15]).

[4]So V could be seen as the cities whereas E could be seen as the routes between cities.

[5]For a discussion of sincere preferences and approval voting in general refer to Brams and Fishburn [4].

2.1 Social Choice Functions

We will use two different concepts of social aggregation.[6] First, a social welfare function F is a function $F : \mathcal{P} \to \mathcal{B}$. Second, a social choice rule f is a function $f : K \times \mathcal{P} \to K$, where K denotes the set of all nonempty and proper subsets of E and for all $S \in K$ and $P \in \mathcal{P}$, $f(S, P) \subseteq S$. We will also distinguish between F and f by saying that F is the global rule, i.e. individual preferences are defined over E, whereas f is the local rule, i.e. individual preferences are defined over some set $T \subset E$. This allows for the following definitions of local and global voting rules:

Definition 1 *Given $\succ_i |T$, assign $|T| - 1$ points to individual i's top element, $|T| - 2$ points to its second ranked element , etc. For any $e \in T$, we say that individual i's Borda count of e is given by $B_i(e) = |\{a \in T : e \succ_i a\}|$. The total Borda count of element e, $B(e) = \sum_1^m B_i(e)$, is determined by summing those scores over all $i \in I$. Now, given $T \subseteq E$ and $P \in \mathcal{P}$, voting rule f_B is the local Borda Rule if for all $e_j \in T$, $e_j \in f_B(T, P)$ if and only if $B(e_j) \geq B(e_k)$ for all $e_k \in T$. In addition, F_B is the global Borda rule if for all $e_j, e_k \in T$, $e_j \succsim_B e_k$ if and only if $B(e_j) \geq B(e_k)$.*

Definition 2 *Given $T \subseteq E$ and $P \in \mathcal{P}$, voting rule f_A is local Approval voting if for all $e_j \in T$, $e_j \in f_A(T, P)$ if and only if $|\{i \in I : e_j \in S_i\}| \geq |\{i \in I : e_k \in S_i\}|$ for all $e_k \in T$. In addition, F_A is global Approval voting if for all $e_j, e_k \in T$, $e_j \succsim_A e_k$ if and only if $|\{i \in I : e_j \in S_i\}| \geq |\{i \in I : e_k \in S_i\}|$.*

Definition 3 *Given $T \subseteq E$ and $P \in \mathcal{P}$, voting rule f_{Pl} is the local Plurality Rule if for all $e_j \in T$, $e_j \in f_{Pl}(T, P)$ if and only if $|\{i \in I : e_j \in S_i^t\}| \geq |\{i \in I : e_k \in S_i^t\}|$ for all $e_k \in T$. In addition, F_{Pl} is the global Plurality rule if for all $e_j, e_k \in T$, $e_j \succsim_{Pl} e_k)$ if and only if $|\{i \in I : e_j \in S_i^t\}| \geq |\{i \in I : e_k \in S_i^t\}|$.*

Definition 4 *Given $T \subseteq E$ and $P \in \mathcal{P}$, voting rule f_M is the local Simple Majority rule if for all $e_j \in T$, $e_j \in f_M(T, P)$ if and only if $p_{j,k} \geq 0$ for all $e_k \in T$. In addition F_M is the global Simple Majority rule if for all $e_j, e_k \in T$, $e_j \succsim_M e_k$ if and only if $p_{j,k} \geq 0$.*

We do not explicitly discuss the issue of ties which could occur in choice sets and/or social rankings. Such ties could for example be broken randomly. Note that the examples discussed in this paper are unambiguous and do not rely on any specific tie-breaking device. The Simple Majority rule though requires special care since there might in principle be preference cycles and/or empty choice sets. Ways to overcome this problem will be discussed in section 3.3.

2.2 Tours

The *traveling salesman problem* (more recently referred to as the traveling salesperson problem) (TSP) is one of the classical pillars of Combinatorial Optimization with two famous books [12], [9] devoted entirely to its study. It can be defined as follows.

[6]This is of importance as global decisions will depend on a ranking of the edges, whereas local decisions only depend on sets of edges.

Definition 5 *Given a graph $G = (V, E)$, a* tour *L is an ordered list of all objects in V, where w can be a direct successor of v in the ordered list only if $\{v, w\} \in E$.*

Every tour L is a solution of TSP. Note that in our case the condition on $\{v, w\}$ will be fulfilled by default, since we assume the graph to be complete. Moreover, every tour L induces in a natural way an associated set of edges T: W.l.o.g. let $L = [v_1, v_2, \ldots, v_k]$ (after renumbering). Then the tour can be described equivalently by the set of k edges

$$T = \{\ \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}\ \}.$$

Obviously, given a starting vertex, from any tour represented by its edge set it is straightforward to reconstruct the corresponding ordered list of objects. Note that if a set of edges describes a tour, then the ordering of these edges is obvious and there is no need to stick to an ordered list. If the objects are cities and the edges are street connections, the original context of a traveling salesman journey arises.

In classical optimization every edge $e \in E$ is assigned a positive cost value and the objective of the optimization is to find a tour with minimal total edge cost. In this paper we abandon this concept in favour of a collective decision and evaluation process. This opens the possibility to compare different tours with respect to dominance concepts. Although the widely used concept of Pareto dominance is too strong for our purpose, we will be able to use one weaker (but still reasonably strong) definition of dominance and one rather weak concept of superiority for our comparisons of different tours. The former is related to what is used in [17], the latter is in the spirit of the usual numerical optimization idea used in the TSP.

Definition 6 *Tour T* pairwise dominates tour T' if and only if for all $i \in I$, there exists a one-to-one function $g_i : T \rightarrow T'$ such that for all $a \in T$, $a \succ_i g_i(a)$.

Definition 7 *For any set $S \subseteq E$, let $B(S) = \sum_{e \in S} B(e)$, denote the Borda count of a whole set of edges. Tour T is strictly* Borda-superior over tour T' if and only if $B(T) > B(T')$.

3 Tours from Collective Decisions

In classical optimization it is well known that the problem of finding a tour with minimal total cost is algorithmically intractable (in fact it is a so-called $\mathcal{NP}$-hard problem[7]). However, the aim of this paper is not the construction of the "best" tour under some concept of dominance but the analysis of different rules to construct a tour given the preferences of a group of individuals. We will stick to the most natural construction rules to maintain the relation to a real-world decision scenario. The Local Greedy Rule constructs a tour by iteratively extending the ordered list of objects whereas the Global Greedy Rule builds up a set of edges which finally represent a tour.

[7] Roughly speaking, this means that all known algorithms, which solve the problem to optimality, may require an amount of running time which increases dramatically (faster than polynomially) when the size of the problem increases. A thorough introduction to computational complexity is given in [7].

Definition 8 Local Greedy Rule: *Having determined a partial solution L starting with v_1 and ending with v_ℓ choose the next object $v_{\ell+1}$ by a local voting rule among all edges $e = \{v_\ell, v_j\}$ with $v_j \notin L$. The final edge to close the tour and return to v_1 is given by default.*

Definition 9 Global Greedy Rule: *Set $T = \emptyset$. Apply a global voting rule to sort E into an ordered list of edges $F = \{e_1, e_2, \ldots\}$. Go through the edges in this order and add an edge $e = \{v, w\}$ to T if each object v and w appears at most once as an endpoint of an edge in the current set T and if e does not induce a cycle in T.*

Note that during the application of the latter rule the set T may well contain unconnected segments of a tour. The final edge to close the tour follows by default.

Recall that in our definition we assume the given graph to be complete. This guarantees that both Greedy rules will always end up with a feasible solution. If the underlying graph is incomplete, i.e. E does not contain all two-element subsets of V, even the problem of finding any tour, more precisely the decision problem whether a tour exists, is algorithmically intractable (an $\mathcal{NP}$-complete decision problem). However, we might well extend such a graph to a complete one by adding edges and inserting them at the bottom of all the indivual rankings. It should be noted that such an extension may sometimes yield solutions which are infeasible for the original problem.

Both algorithms are well-known in classical combinatorial optimization. The local greedy rule is also known as *Nearest Neighbor* heuristic which moves from every visited city to its nearest neighbor by a local decision. The global greey rule, which sorts the edges in increasing order of length, is sometimes referred to as the *multi-fragment heuristic* (cf. Bentley [3]). If the distances resp. costs between cities obey the triangle inequality then it was shown that the solutions derived by the two algorithms can not be arbitrarily bad but both have a total value at most $\log n$ times[8] the optimal solution value (see Rosenkrantz et al. [20] resp. Ong and Moore [16]). However, there were also instances constructed where the solutions can be close to this upper bound.

An intuitive assessment might indicate that both rules for collective decisions should perform reasonably well with a bias towards the global rule. However we can show that for all the voting rules under consideration both rules may produce tours of low approval. In particular for the Plurality rule, the Borda rule and Approval voting, there exist lists of individual preferences such that the local tour is pairwise dominated by a disjoint tour, which - maybe to some surprise - is the global tour. For the latter two rules and different individual preferences this will also work vice versa. So it seems that whatever reasonable voting rule we consider for our group decision process, there could occur situations in which tours completely different from those derived via a greedy rule, make all individuals better off.

3.1 Borda Rule

To facilitate understanding of the main principle which carries through most of the constructions in this paper we start with the following example of 5 vertices.

[8] The precise value of this worst-case upper bound is $\frac{1}{2}(\lceil \log_2 n \rceil + 1)$.

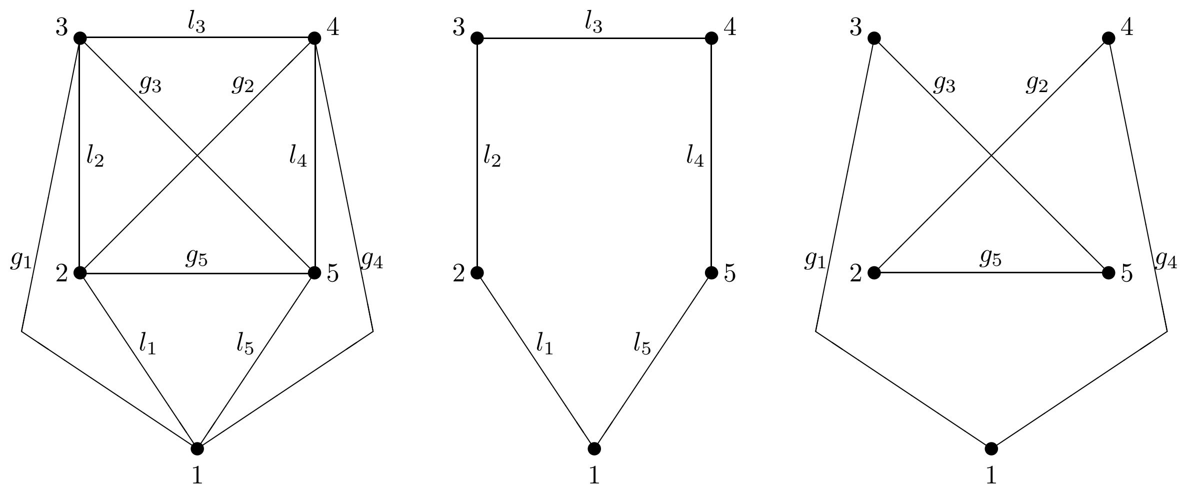

Figure 1: Example: Complete graph with $n = 5$ vertices (left), solution derived by Local Greedy Rule (center) and solution given by Global Greedy Rule (right) for both examples.

Example: Consider a complete graph with 5 vertices and 10 edges (known as the K_5 graph) with the notation as given in the left picture in Figure 1, i.e. $E = \{l_1, l_2, ..., l_5, g_1, g_2, ..., g_5\}$, where edges l_i and g_i both have vertex i in common. In the example we have 4 individuals with strict rankings over the set of edges as given in Table 1 (columns 2 to 5). Column 1 indicates the Borda scores given to the edges in the respective position in the individual rankings. Column 6 and 7 present the edges and their total Borda counts. Observe the periodicity in the ranking of the l and g edges.

Borda score	1	2	3	4	edge	Borda count
9	$\mathbf{g_1}$	$\mathbf{g_2}$	$\mathbf{g_3}$	$\mathbf{g_4}$	l_1	20
8	l_2	l_3	l_1	l_4	l_2	20
7	$\mathbf{g_2}$	$\mathbf{g_3}$	$\mathbf{g_4}$	$\mathbf{g_1}$	l_3	20
6	l_3	l_1	l_4	l_2	l_4	20
5	$\mathbf{g_3}$	$\mathbf{g_4}$	$\mathbf{g_1}$	$\mathbf{g_2}$	l_5	0
4	l_1	l_4	l_2	l_3	$\mathbf{g_1}$	24
3	$\mathbf{g_4}$	$\mathbf{g_1}$	$\mathbf{g_2}$	$\mathbf{g_3}$	$\mathbf{g_2}$	24
2	l_4	l_2	l_3	l_1	$\mathbf{g_3}$	24
1	$\mathbf{g_5}$	$\mathbf{g_5}$	$\mathbf{g_5}$	$\mathbf{g_5}$	$\mathbf{g_4}$	24
0	l_5	l_5	l_5	l_5	$\mathbf{g_5}$	4

Table 1: Example: Borda - local tour pairwise dominated by global tour (in bold).

Let us first consider the local greedy solution and assume that the selection starts at vertex 1. The set of feasible edges at vertex 1 is given by $E_1 = \{l_1, l_5, g_1, g_4\}$ and hence the preference profile P is restricted to $P|E_1$, which is presented in Table 2.

Borda score	1	2	3	4	edge	Borda count
3	g_1	l_1	l_1	g_4	l_1	9
2	l_1	g_4	g_4	g_1	l_5	0
1	g_4	g_1	g_1	l_1	g_1	7
0	l_5	l_5	l_5	l_5	g_4	8

Table 2: Example: Borda scores for restricted profile at vertex 1.

Obviously the local decision taken by the group at vertex 1 based on the Borda count will be to move along edge l_1 to vertex 2. At vertex 2, the set of feasible edges reduces to $E_2 = \{l_2, g_2, g_5\}$ and the preference profile $P|E_2$ is stated in Table 3.

Borda score	1	2	3	4	edge	Borda count
2	l_2	g_2	l_2	l_2	l_2	7
1	g_2	l_2	g_2	g_2	g_2	5
0	g_5	g_5	g_5	g_5	g_5	0

Table 3: Example: Borda scores for restricted profile at vertex 2.

From the above table we observe that the group now moves along edge l_2 to vertex 3, where $E_3 = \{l_3, g_3\}$ and the new restricted profile is $P|E_3$. The decision now reduces to a pairwise majority decision between edges l_3 and g_3 which the former wins by a 3 vs. 1 tally taking the tour to vertex 4. By definition of a tour, only vertex 5 remains to be visited which is possible only by moving along edges l_4 and l_5 back to the starting point at vertex 1. No further decisions are required for this step. Hence the tour determined by the local greedy rule is $T_\ell = \{l_1, l_2, l_3, l_4, l_5\}$ (middle picture in Figure 1).

Considering the global greedy rule we see from Table 1 that there are 4 edges with the same largest total Borda count of 24. These are g_1, g_2, g_3 and g_4. As choosing all of them is consistent with a single tour, this - together with edge g_5 - determines the global tour, i.e. $T_g = \{g_1, g_2, g_3, g_4, g_5\}$ (right picture in Figure 1).

How are these two tours related to each other with respect to the individual preferences? Obviously not all edges in T_g are strictly preferred to those in T_ℓ by all individuals as can be seen in Table 1. However, there is pairwise domination, i.e. for every individual there is a one-to-one function that assigns an edge in T_ℓ to any edge in T_g such that the latter is preferred to the former by that individual.

This example can be extended to complete graphs of any size such that $|V| \geq 5$. The result is stated in the following proposition.

Proposition 1 *Using the Borda rule, for any complete graph with $|V| \geq 5$, there exist preference profiles such that the local tour is pairwise dominated by the global tour.*

Proof. We will split the proof into two parts, one for odd and one for even numbers of vertices. Let us first start with odd numbers of vertices, $|V| \geq 5$. We will partition the set E into 3 distinct sets. The first set L contains all edges connecting vertices with

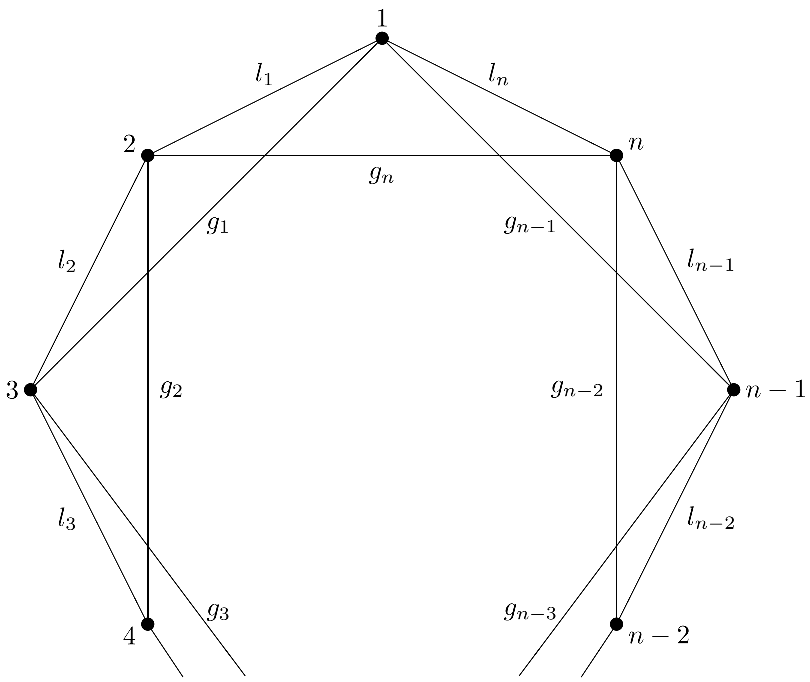

Figure 2: General setup of a graph with an odd number of $|V| = n$ vertices and edge sets $L = \{l_1, \ldots, l_n\}$ and $G = \{g_1, \ldots, g_n\}$.

consecutive numbers, i.e. the set $L = \{l_1, l_2, ..., l_n\}$ with $l_i = \{v_i, v_{i+1}\}$ and $l_n = \{v_n, v_1\}$. The second set $G = \{g_1, g_2, ..., g_n\}$ contains edges with $g_i = \{v_i, v_{i+2}\}$, $g_{n-1} = \{v_{n-1}, v_1\}$ and $g_n = \{v_n, v_2\}$. All other edges in the graph are denoted as set X. A sketch of the graph is given in Figure 2. Note that both L and G determine a tour for an odd number of vertices. Consider now $n-1$ individuals with preferences as given in Table 4 where an edge being above (below) the set X means that this edge is strictly preferred to (by) all edges in X.

Let us determine the global tour first. W.l.o.g. we assign a value of 0 to all elements of X. An elementary calculation yields the Borda counts[9] of all edges given in Table 4. The values imply immediately that the global tour is $T_g = G$, since $n^2 + n - 2 > n^2 - 1$ for $n > 1$.

Now consider the local decisions necessary to determine the local tour. At vertex 1 the group has to decide on the set E_1 with $X_1 := X \cap E_1$ and $\{l_1, l_n, g_1, g_{n-1}\} \subset E_1$. The restricted profile $P|E_1$ is stated in Table 5. As can be easily seen, l_1 will get the highest Borda count for $n \geq 5$.

Hence the group moves along l_1 to vertex 2 where the decision has to be taken between edges in $E_2 = \{l_2, g_2, g_n\} \cup X_2$, where $X_2 \subseteq X$ denotes all edges between vertex 2 and a vertex that has not been passed yet, excluding $\{l_2, g_2, g_n\}$. The restricted profile at vertex 2 and the Borda count of the edges is given in Table 6. Obviously, l_2 has the highest Borda count for $n \geq 5$ and the group moves along l_2 to vertex 3.

Starting with vertex 3, any local decision will be between l_i, g_i and all edges in $X_i := \{\{i, j\} \mid j > i\} \cap X$ for $i \geq 3$. Table 4 we observe that l_i is above g_i in $n-2$ rankings and below g_i in only one ranking. As all edges in X_i are always lower ranked

[9]What we call Borda count is of course not the real Borda count as we neglect the edges in X. However, the ranking of the edges under investigation is not influenced by this convention.

Borda score	1	2	3	...	$n-1$	edge	Borda count
$2n$	$\mathbf{g_1}$	$\mathbf{g_2}$	$\mathbf{g_3}$	...	$\mathbf{g_{n-1}}$	l_1	n^2-1
$2n-1$	l_2	l_3	l_4	...	l_{n-1}	l_2	n^2-1
$2n-2$	$\mathbf{g_2}$	$\mathbf{g_3}$	$\mathbf{g_4}$	...	$\mathbf{g_1}$	$\vdots$	$\vdots$
$2n-3$	l_3	l_4	l_5	...	l_2	$\vdots$	$\vdots$
$2n-4$	$\mathbf{g_3}$	$\mathbf{g_4}$	$\mathbf{g_5}$	...	$\mathbf{g_2}$	l_{n-1}	n^2-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	l_n	$n-1$
7	l_{n-2}	l_1	l_{n-1}	...	l_{n-3}	$\mathbf{g_1}$	n^2+n-2
6	$\mathbf{g_{n-2}}$	$\mathbf{g_{n-1}}$	$\mathbf{g_1}$	...	$\mathbf{g_{n-3}}$	$\mathbf{g_2}$	n^2+n-2
5	l_1	l_{n-1}	l_2	...	l_{n-2}	$\vdots$	$\vdots$
4	$\mathbf{g_{n-1}}$	$\mathbf{g_1}$	$\mathbf{g_2}$	...	$\mathbf{g_{n-2}}$	$\vdots$	$\vdots$
3	l_{n-1}	l_2	l_3	...	l_1	$\vdots$	$\vdots$
2	$\mathbf{g_n}$	$\mathbf{g_n}$	$\mathbf{g_n}$	...	$\mathbf{g_n}$	$\mathbf{g_{n-1}}$	n^2+n-2
1	l_n	l_n	l_n	...	l_n	$\mathbf{g_n}$	$2n-2$
0	X	X	X	...	X	X	0

Table 4: Borda - local tour pairwise dominated by global tour (in bold). All elements in X are given a Borda score of 0.

Borda score	1	2	3	...	$n-1$	edge	Borda count
4	g_1	l_1	l_1	...	g_{n-1}	l_1	$4n-7$
3	l_1	g_{n-1}	g_{n-1}	...	g_1	l_n	$n-1$
2	g_{n-1}	g_1	g_1	...	l_1	g_1	$2n+1$
1	l_n	l_n	l_n	...	l_n	g_{n-1}	$3n-3$
0	X_1	X_1	X_1	...	X_1	X_1	0

Table 5: Borda - restricted profile at vertex 1.

than both, l_i and g_i, the former will be the Borda winner at every such vertex. This determines our local tour which is $T_\ell = \{l_1, l_2, ..., l_n\}$ and therefore disjoint from our global tour T_g which it dominates pairwise as can be seen in Table 4.

Let us now turn to graphs with an even number of edges. An example for $|V| = 6$ is given in Figure 3. Note that the labelling of the edges is slightly different from the odd case in the sense that a g_i-edge does not always connect vertex i with $i+2$. The pattern for the even case is the following: g_{n-1} connects vertex 1 with vertex 4 and g_2 connects vertex 2 with vertex $n-1$.

For all other edges the same notation as in the odd case applies with g_i connecting vertices i and $i+2$ and g_n connecting vertices n and 2. All l_i edges again connect vertex i with vertex $i+1$, and the remaining edges are collected in set X. Relabeling the edges in this way enables us to use again the profile given in Table 4 to establish the desired result. All decision situations are identical to the case of an odd number of vertices. ■

This result however cannot be used to show a certain kind of superiority of the

Borda score	1	2	3	...	$n-1$	edge	Borda count
3	l_2	g_2	l_2	...	l_2	l_2	$3n-4$
2	g_2	l_2	g_2	...	g_2	g_2	$2n-1$
1	g_n	g_n	g_n	...	g_n	g_n	$n-1$
0	X_2	X_2	X_2	...	X_2	X_2	0

Table 6: Borda - restricted profile at vertex 2.

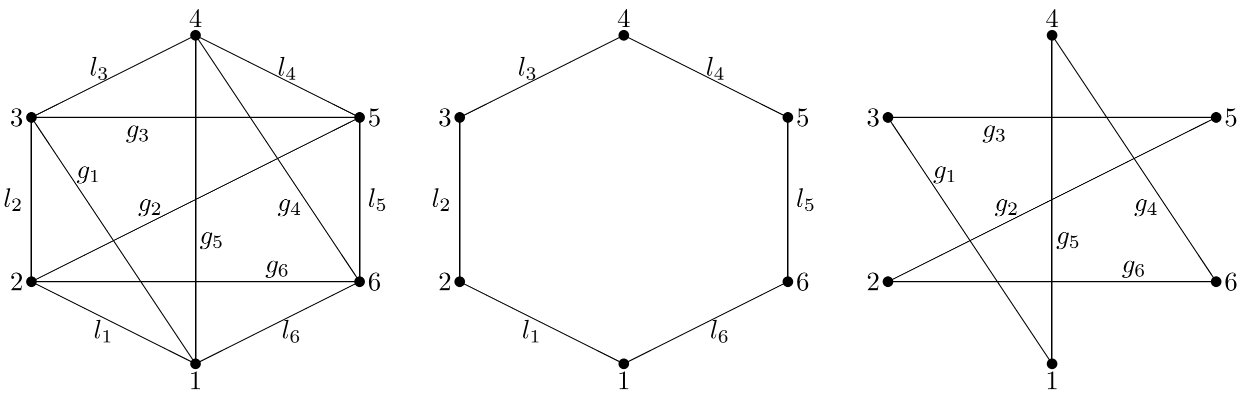

Figure 3: Example: Graph with $n = 6$ vertices (left, irrelevant edges $\{1,5\},\{2,4\},\{3,6\}$ are not shown), solution derived by Local Greedy Rule (center) and solution given by Global Greedy Rule (right) for both examples.

global rule as we can also find preference profiles for which the exact opposite holds. Again we start with an easier example before moving on to a general statement.

Example: Consider again the graph shown in Figure 1. Now the preference profile over the set of edges is given in Table 7, where we consider 5 individuals.

Let us start with the global tour. Ranking the alternatives according to the Borda count (see the final column in Table 7) has g_2 above g_5 above g_3. However, those three edges determine the remaining two edges needed for the global tour, namely edges g_1 and g_4. Hence the global tour is $T_g = \{g_1, g_2, g_3, g_4, g_5\}$.

Applying the local greedy rule starting at vertex 1, the first decision to be taken will be among the set of edges $E_1 = \{l_1, l_5, g_1, g_4\}$. The restricted preference profile and the respective Borda counts are presented in Table 8. Obviously, the local decision taken by the group at vertex 1 based on the Borda count will be to move along edge l_1 to vertex 2. At vertex 2, the set of feasible edges reduces to $E_2 = \{l_2, g_2, g_5\}$ and the corresponding preference profile $P|E_2$ is stated in Table 9.

We observe that the group now moves along edge l_2 to vertex 3, where $E_3 = \{l_3, g_3\}$. This now reduces to a pairwise majority decision between edges l_3 and g_3 with the former being preferred over the latter by a tally of 3 vs. 2. Hence, the group moves along edge l_3 to vertex 4. The two necessary edges to complete the tour are again edges l_4 and l_5 and the tour determined by the local greedy rule is $T_\ell = \{l_1, l_2, l_3, l_4, l_5\}$. Comparing T_g with T_ℓ in the original preference profile of Table 7, we see that the former tour is dominated by the latter with respect to pairwise dominance.

Borda score	1	2	3	4	5	edge	Borda count	Borda rank
9	$\mathbf{l_2}$	$\mathbf{l_1}$	$\mathbf{l_3}$	$\mathbf{l_5}$	$\mathbf{l_5}$	$\mathbf{l_1}$	27	(4)
8	g_5	g_3	g_2	g_5	g_2	$\mathbf{l_2}$	27	(4)
7	$\mathbf{l_3}$	$\mathbf{l_2}$	$\mathbf{l_1}$	$\mathbf{l_4}$	$\mathbf{l_4}$	$\mathbf{l_3}$	27	(4)
6	g_2	g_5	g_3	g_2	g_3	$\mathbf{l_4}$	23	(7)
5	$\mathbf{l_1}$	$\mathbf{l_3}$	$\mathbf{l_2}$	$\mathbf{l_3}$	$\mathbf{l_2}$	$\mathbf{l_5}$	21	(8)
4	g_3	g_2	g_5	g_3	g_5	g_1	10	(9)
3	$\mathbf{l_4}$	$\mathbf{l_4}$	$\mathbf{l_4}$	$\mathbf{l_1}$	$\mathbf{l_1}$	g_2	32	(1)
2	g_1	g_1	g_1	g_1	g_1	g_3	28	(3)
1	$\mathbf{l_5}$	$\mathbf{l_5}$	$\mathbf{l_5}$	$\mathbf{l_2}$	$\mathbf{l_3}$	g_4	0	(10)
0	g_4	g_4	g_4	g_4	g_4	g_5	30	(2)

Table 7: Example: Borda - global tour pairwise dominated by local tour (in bold).

Borda score	1	2	3	4	5	edge	Borda count	Borda rank
3	l_1	l_1	l_1	l_5	l_5	l_1	13	(1)
2	g_1	g_1	g_1	l_1	l_1	l_5	9	(2)
1	l_5	l_5	l_5	g_1	g_1	g_1	8	(3)
0	g_4	g_4	g_4	g_4	g_4	g_4	0	(4)

Table 8: Example: Borda - restricted profile at vertex 1.

Borda score	1	2	3	4	5	edge	Borda count	Borda rank
2	l_2	l_2	g_2	g_5	g_2	l_2	6	(1)
1	g_5	g_5	l_2	g_2	l_2	g_2	4	(3)
0	g_2	g_2	g_5	l_2	g_5	g_5	5	(2)

Table 9: Example: Borda - restricted profile at vertex 2.

Proposition 2 *Using the Borda rule, for any complete graph with $|V| \geq 5$, there exist preference profiles such that the global tour is pairwise dominated by the local tour.*

Proof. As before we will split the proof into two parts, one for odd and one for even numbers of vertices and start with the former with $|V| > 5$. The case $|V| = 5$ was shown in the above example. As before, we will partition the set E into 3 distinct sets L, G and X. Consider now $n-1$ individuals with preferences as given in Table 10 where an edge being above (below) the set X means that this edge is strictly preferred to (by) all edges in X.

Let us determine the global tour first. Calculating the Borda count for this preference profile with symmetric character, there is

$$B(l_i) = \sum_{k=1}^{n-2}(x+3+2k)+1 = (x+3)(n-2)+2\sum_{k=1}^{n-2}k+1 = (n-2)(n+x+2)+1$$

for $i = 1, 2, \ldots, n-1$ whereas $B(l_n) = (n-1)(x+3)$. On the other hand it is easy to

Borda score	1	2	3	...	$n-2$	$n-1$
$2n+x-1$	$\mathbf{l_2}$	$\mathbf{l_3}$	$\mathbf{l_4}$	...	$\mathbf{l_1}$	$\mathbf{l_{n-1}}$
$2n+x-2$	g_1	g_2	g_3	...	g_{n-2}	g_{n-1}
$2n+x-3$	$\mathbf{l_3}$	$\mathbf{l_4}$	$\mathbf{l_5}$	...	$\mathbf{l_{n-1}}$	$\mathbf{l_2}$
$2n+x-4$	g_2	g_3	g_4	...	g_{n-1}	g_1
$2n+x-5$	$\mathbf{l_4}$	$\mathbf{l_5}$	$\mathbf{l_6}$	...	$\mathbf{l_2}$	$\mathbf{l_3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x+6$	g_{n-3}	g_{n-2}	g_{n-1}	...	g_{n-5}	g_{n-4}
$x+5$	$\mathbf{l_1}$	$\mathbf{l_{n-1}}$	$\mathbf{l_2}$	...	$\mathbf{l_{n-3}}$	$\mathbf{l_{n-2}}$
$x+4$	g_{n-2}	g_{n-1}	g_1	...	g_{n-4}	g_{n-3}
$x+3$	$\mathbf{l_n}$	$\mathbf{l_n}$	$\mathbf{l_n}$	...	$\mathbf{l_n}$	$\mathbf{l_n}$
$x+2$	g_{n-1}	g_1	g_2	...	g_{n-3}	g_{n-2}
$\vdots$	X	X	X	...	X	X
1	$\mathbf{l_{n-1}}$	$\mathbf{l_2}$	$\mathbf{l_3}$	...	$\mathbf{l_{n-2}}$	$\mathbf{l_1}$
0	g_n	g_n	g_n	...	g_n	g_n

Table 10: Borda - global tour pairwise dominated by local tour (in bold). $x := |X|$

compute that

$$B(g_i) = \sum_{k=1}^{n-1}(x+2k) = x(n-1) + 2\sum_{k=1}^{n-1} k = (n-1)(n+x)$$

for $i = 1, 2, \ldots, n-1$ and $B(g_n) = 0$. By elementary algebra we get for $i = 1, \ldots, n-1$

$$B(l_i) < B(g_i) \Longleftrightarrow n < x+3.$$

For a complete graph there is $x = \binom{n}{2} - 2n$ and the above inequality holds for $n \geq 7$. Naturally, all other edges also have a lower Borda count than g_1. This implies that the global tour is $T_g = G$.

Now consider the local decisions necessary to determine the local tour. At vertex 1 the group has to decide on the set $E_1 = \{l_1, l_n, g_1, g_{n-1}\} \cup X_1$ defined as before. The restricted profile $P|E_1$ is stated in Table 11.

Borda score	1	2	3	...	$n-2$	$n-1$
x_1+3	g_1	l_1	l_1	...	l_1	g_{n-1}
x_1+2	l_1	g_{n-1}	g_{n-1}	...	g_{n-1}	g_1
x_1+1	l_n	l_n	g_1	...	g_1	l_n
$\vdots$	g_{n-1}	g_1	l_n	...	l_n	X_1
$\vdots$	X_1	X_1	X_1	...	X_1	l_1

Table 11: Borda - restricted profile at vertex 1. $x_1 := |X_1|$

Evaluating the local Borda counts and plugging in $x_1 = n-5$ (on a complete graph) we get $B(l_1) = (n-3)(x_1+3) + x_1 + 2 = n^2 - 4n + 3$, $B(l_n) = (n-4)x_1 +$

$3x_1 + 3 = n^2 - 6n + 8$, $B(g_1) = (n-4)(x_1+1) + 3x_1 + 5 = n^2 - 5n + 6$ and $B(g_{n-1}) = (n-3)(x_1+2) + 2x_1 + 3 = n^2 - 4n + 2$. Hence the group moves along l_1 to vertex 2 where the decision has to be taken between edges in $E_2 = \{l_2, g_2, g_n\} \cup X_2$, where X_2 is defined as before. The restricted profile at vertex 2 is given in Table 12.

Borda score	1	2	3	...	$n-2$	$n-1$
$x_2 + 2$	l_2	g_2	l_2	...	l_2	l_2
$\vdots$	g_2	X_2	g_2	...	g_2	g_2
$\vdots$	X_2	l_2	X_2	...	X_2	X_2
0	g_n	g_n	g_n	...	g_n	g_n

Table 12: Borda - restricted profile at vertex 2. $x_2 := |X_2|$

Since $x_2 = n-5$,[10] the local Borda counts are given by $B(l_2) = (n-2)(x_2+2)+1 = n^2 - 5n + 7$, $B(g_2) = (n-2)(x_2+1) + x_2 + 2 = n^2 - 5n + 5$ and $B(g_n) = 0$. It follows immediately that the group moves along l_2 to vertex 3.

Beginning with vertex 3, any local decision will be between l_i, g_i and all edges in X_i for $i \geq 3$. From Table 10 we observe that l_i is above g_i in $n-2$ rankings and below g_i in only one ranking by a maximum of $n-3$ positions. Hence, l_i will be the Borda winner at every such vertex. This determines our local tour as $T_\ell = \{l_1, l_2, ..., l_n\}$ which is therefore disjoint from our global tour T_g and dominates it pairwise as can be seen in Table 10.

Turning now to graphs with an even number of edges we refer again to Figure 3 for $|V| = 6$. With exactly the same relabeling of edges as in the proof of Proposition 1, we can again use the profile given in Table 10 to establish the desired result for $|V| \geq 8$. All decision situations are identical to the case with an odd number of vertices in that respect. This leaves open the case $|V| = 6$ for which we consider the preference profile in Table 13 in which $X = \{(1,5), (2,4), (3,6)\}$ is the set of all edges which are not explicitly stated. We leave the derivation of the respective tours to the reader.

■

It should be noted that the fact that a tour pairwise dominates its complement, does not imply that it is in some sense the best possible tour. Indeed, the following can be shown:

Proposition 3 *There exist preference profiles such that even if one of the two greedy rules using the Borda rule pairwise dominates the other, there exists a third tour which is Borda-superior to both.*

Proof. Following the definition of a graph with an odd number of n vertices depicted in Figure 2 consider the set of edges

$$T^* := \{g_1, g_3, \dots, g_{n-2}, l_{n-1}, g_{n-3}, g_{n-5}, \dots, g_2, l_1\}.$$

It is easy to see that T^* is a tour.

[10] Note that X_2 does not containt edges to vertices $1, 2, 3, 4, n$.

Borda score	1	2	3	4	5
14	$\mathbf{l_6}$	$\mathbf{l_3}$	$\mathbf{l_4}$	$\mathbf{l_1}$	$\mathbf{l_5}$
13	g_1	g_2	g_3	g_4	g_5
12	$\mathbf{l_2}$	$\mathbf{l_4}$	$\mathbf{l_1}$	$\mathbf{l_5}$	$\mathbf{l_2}$
11	g_2	g_3	g_4	g_5	g_1
10	$\mathbf{l_3}$	$\mathbf{l_1}$	$\mathbf{l_5}$	$\mathbf{l_2}$	$\mathbf{l_3}$
9	g_3	g_4	g_5	g_1	g_2
8	$\mathbf{l_4}$	$\mathbf{l_6}$	$\mathbf{l_2}$	$\mathbf{l_3}$	$\mathbf{l_4}$
7	g_4	g_5	g_1	g_2	g_3
6	$\mathbf{l_1}$	$\mathbf{l_5}$	$\mathbf{l_6}$	$\mathbf{l_6}$	$\mathbf{l_6}$
5	g_5	g_1	g_2	g_3	g_4
$\vdots$	X	X	X	X	X
1	$\mathbf{l_5}$	$\mathbf{l_2}$	$\mathbf{l_3}$	$\mathbf{l_4}$	$\mathbf{l_1}$
0	g_6	g_6	g_6	g_6	g_6

Table 13: Example $|V| = 6$: Borda - global tour pairwise dominated by local tour (in bold).

In the example of Table 4, the global tour pairwise dominates the local tour and has a total Borda Count $B(T_g) = (n-1)(n^2+n-2)+2n-2$. Comparing T_g with T^* we get $B(T^*) - B(T_g) = 2(n^2-1) - (n^2+n-2+2n-2) = n^2-3n+2$, which is positive for $n \geq 5$.

In the example of Table 10, the local tour pairwise dominates the global tour. Note that $B(T_\ell) = B(T_g) + (n-1)n$ since each of the $n-1$ individuals ranks each edge in T_ℓ exactly one position higher than a corresponding edge in T_g. The comparsion of T_ℓ and T^* yields (plugging in $x = n(n-1)/2 - 2n$)

$$\begin{aligned}
B(T^*) - B(T_\ell) &= B(T_g) + B(l_1) + B(l_{n-1}) - B(g_{n-1}) - B(g_n) - B(T_g) - (n-1)n \\
&= 2((n-2)(n+x+2)+1) - (n-1)(n+x) - (n-1)n \\
&= nx - 3x + 2n - 6 \\
&= n^3 - 8n^2 + 19n - 12,
\end{aligned}$$

which is positive for $n \geq 7$. For an even number of edges we need a slightly different adaptation of the global tour in the following way:

$$T^* := \{l_1, l_2, g_3, g_5, \ldots, g_{n-3}, l_{n-1}, g_{n-2}, g_{n-4}, \ldots, g_6, g_4, g_{n-1}\}.$$

That T^* has a higher Borda count than both, T_g and T_l in Table 4 as well as in Table 10 is left to the reader.

The case $n = 5$ can be settled easily by considering the corresponding example of Table 7 and setting $T^* = \{l_1, g_2, l_3, g_3, l_5\}$ which yields $B(T^*) = 135$, whereas $B(T_\ell) = 125$.

Note that in both cases of this proposition T^* is Borda-optimal in the sense of classical optimization. ∎

The question now arises whether mutual dominance of local and global decisions is an artefact of the Borda rule and probably other scoring rules. As we will show, this is not the case as the same results can occur with another well known voting procedure, namely Approval voting.

3.2 Approval Voting

Proposition 4 *Using Approval Voting, for any complete graph with* $|V| \geq 5$*, there exist preference profiles such that the local tour is pairwise dominated by the global tour.*

Proof. Consider again a graph with an odd number of vertices as shown in Figure 1 and individual preferences of 7 voters over the set of edges as given in Table 14. All edges are as previously defined, i.e. $g_i \in G$, $l_i \in L$ and all other edges are in X. The horizontal lines in the rankings determine the border between the approved edges of that individual (those above the line) and those not approved (those below the line).

<table>
<tr><th>1, 2</th><th>3, 4</th><th>5, 6</th><th>7</th></tr>
<tr><td>$\mathbf{g_1}$</td><td>$\mathbf{g_1}$</td><td>$\mathbf{g_{n-1}}$</td><td>$\mathbf{g_1}$</td></tr>
<tr><td>l_1</td><td>l_1</td><td>l_1</td><td>$\mathbf{g_2}$</td></tr>
<tr><td>$\mathbf{g_n}$</td><td>$\mathbf{g_n}$</td><td>$\mathbf{g_2}$</td><td>$\mathbf{g_3}$</td></tr>
<tr><td>l_2</td><td>l_2</td><td>l_n</td><td>$\mathbf{g_4}$</td></tr>
<tr><td>$\mathbf{g_{n-1}}$</td><td>$\mathbf{g_2}$</td><td>X</td><td>⋮</td></tr>
<tr><td>l_3</td><td>l_3</td><td>$\mathbf{g_1}$</td><td style="border-bottom:1px solid">$\mathbf{g_n}$</td></tr>
<tr><td>$\mathbf{g_3}$</td><td>$\mathbf{g_3}$</td><td>l_3</td><td>l_1</td></tr>
<tr><td>l_4</td><td>l_4</td><td>$\mathbf{g_3}$</td><td>l_2</td></tr>
<tr><td>$\mathbf{g_4}$</td><td>$\mathbf{g_4}$</td><td>l_2</td><td>⋮</td></tr>
<tr><td>⋮</td><td>⋮</td><td>$\mathbf{g_n}$</td><td>⋮</td></tr>
<tr><td style="border-bottom:1px solid">$\mathbf{g_{n-2}}$</td><td>⋮</td><td>l_4</td><td>⋮</td></tr>
<tr><td>l_n</td><td>l_n</td><td>$\mathbf{g_4}$</td><td>⋮</td></tr>
<tr><td>X</td><td>X</td><td>⋮</td><td>⋮</td></tr>
<tr><td>$\mathbf{g_2}$</td><td style="border-bottom:1px solid">$\mathbf{g_{n-1}}$</td><td style="border-bottom:1px solid">$\mathbf{g_{n-2}}$</td><td>l_n</td></tr>
<tr><td>l_{n-1}</td><td>l_{n-1}</td><td>l_{n-1}</td><td>X</td></tr>
</table>

Table 14: AV - local tour pairwise dominated by global tour (in bold).

Let us start with the global decision. As can be calculated easily from Table 14, all edges in $G - \{g_2\}$ will get 7 votes, whereas due to individual 7 all other edges will get strictly less than 7 votes. However, this determines the global tour as $T_g = G$.

For the local tour we will start at vertex 1 with the same reduced set of edges $E_1 = \{l_1, l_n, g_1, g_{n-1}\} \cup X_1$ defined as before. The restricted preference profile $P|E_1$ is given in Table 15. Again, lines in the rankings will separate approved from non-approved edges. Note that we will use the following reasonable assumption: if the reduced set of edges contains approved and non-approved edges according to the original ranking, the individual will also separate those edges in the reduced preference. If the reduced set only consists of either approved or non-approved edges, the individual will vote for all but the bottom ranked edges according to its reduced preference.[11]

1, 2	3, 4	5, 6	7	edge	Approval count
g_1	g_1	g_{n-1}	g_1	l_1	6
l_1	l_1	l_1	g_{n-1}	l_n	4
g_{n-1}	l_n	l_n	l_1	g_1	5
l_n	X_1	X_1	l_n	g_{n-1}	5
X_1	g_{n-1}	g_1	X_1	X_1	4

Table 15: AV - restricted profile at vertex 1.

The largest approval count at vertex 1 is obtained by edge l_1 and the group moves along l_1 to vertex 2 where the reduced preference profile and the respective approval counts are as in Table 16.[12]

1, 2	3, 4	5, 6	7	edge	Approval count
g_n	g_n	g_2	g_2	l_2	6
l_2	l_2	l_2	g_n	g_2	5
X_2	g_2	X_2	l_2	g_n	5
g_2	X_2	g_n	X_2	X_2	≤ 4

Table 16: AV - restricted profile at vertex 2.

Again, the largest approval count at vertex 2 is obtained by edge l_2 and therefore the group moves along l_2 to vertex 3 where the decision is in principle reduced to a pairwise majority decision between edges l_3 and g_3. The restricted profile $P|E_3$ is given in Table 17.

1, 2	3, 4	5, 6	7	edge	Approval count
l_3	l_3	X_3	g_3	l_3	6
g_3	g_3	l_3	l_3	g_3	5
X_3	X_3	g_3	X_3	X_3	≤ 4

Table 17: AV - restricted profile at vertex 3.

The unique Approval vote winner at vertex 3 is l_3. Obviously similar situations occur at any vertex $i > 3$ and we therefore conclude that the local tour is $T_\ell = L$. Note that the construction of the proof does not depend on the existence of a nonempty set X and hence remains valid also for $|V| = 5$.

As we have done in our previous proofs, we can also consider graphs with even numbers of vertices such that $|V| \geq 6$. Relabeling the edges as in the proof for the

[11] This makes sense as we assume the individual to emphasize those edges that it (relatively) prefers. It is also what Brams and Fishburn [4] call *admissible strategies.*

[12] Note that for voters 3 and 4 we put the line above X_2 for convenience. Of course it should be clear that if X_2 contains more than one edge, the line would actually be drawn somewhere within set X_2. We keep this simplification also for further tables.

Borda count enables us to use the same profile as in Table 14 and follow the same argumentation as in the odd case.

If we compare the two tours T_g and T_ℓ on the basis of the individual preferences as given in Table 14 we see that the global tour pairwise dominates the local tour and therefore the proof is complete. ■

Similar to the Borda rule, this dominance works also the other way round using Approval voting. We can state the result in the following proposition.

Proposition 5 *Using Approval Voting, for any complete graph with $|V| \geq 5$, there exist preference profiles such that the global tour is pairwise dominated by the local tour.*

Proof.

1	2	3	4	...	$\frac{n}{2}$	$\frac{n}{2}+1$
$\mathbf{l_2}$	$\mathbf{l_1}$	$\mathbf{l_1}$	$\mathbf{l_1}$	...	$\mathbf{l_1}$	$\mathbf{l_1}$
Y_2	Y_1	Y_1	Y_1	...	Y_1	g_1
g_2	g_2	g_n	g_n	...	g_2	$\mathbf{l_2}$
$\mathbf{l_3}$	$\mathbf{l_n}$	$\mathbf{l_3}$	$\mathbf{l_3}$	...	$\mathbf{l_3}$	g_2
Y_3	Y_n	Y_3	Y_3	...	Y_3	$\mathbf{l_3}$
g_3	g_3	g_3	g_3	...	g_3	g_3
$\mathbf{l_4}$	$\mathbf{l_4}$	$\mathbf{l_2}$	$\mathbf{l_4}$	...	$\mathbf{l_4}$	$\vdots$
Y_4	Y_4	Y_2	Y_4	...	Y_4	$\mathbf{l_{n-1}}$
g_4	g_4	g_4	g_4	...	g_4	g_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\mathbf{l_n}$
$\mathbf{l_{n-1}}$	$\mathbf{l_{n-1}}$	$\mathbf{l_{n-1}}$	$\mathbf{l_{n-1}}$	...	$\mathbf{l_n}$	g_{n-1}
Y_{n-1}	Y_{n-1}	Y_{n-1}	Y_{n-1}	...	Y_n	Y_1
g_n	g_n	g_2	g_2	...	g_n	Y_2
$\mathbf{l_1}$	$\mathbf{l_3}$	$\mathbf{l_5}$	$\mathbf{l_7}$	...	$\mathbf{l_{n-1}}$	Y_3
Y_1	Y_3	Y_5	Y_7	...	Y_{n-1}	$\vdots$
g_{n-1}	g_{n-1}	g_{n-1}	g_{n-1}	...	g_{n-1}	$\vdots$
$\mathbf{l_n}$	$\mathbf{l_2}$	$\mathbf{l_4}$	$\mathbf{l_6}$	...	$\mathbf{l_{n-2}}$	$\vdots$
Y_n	Y_2	Y_4	Y_6	...	Y_{n-2}	Y_{n-1}
g_1	g_1	g_1	g_1	...	g_1	Y_n

Table 18: AV - global tour pairwise dominated by local tour (in bold).

Consider the preference profile given in Table 18, where $Y_i := \{\{i,j\} \mid j > i\} - \{l_k, g_k \mid k = 1, \ldots, n\}$ for $i = 1, \ldots, n$.[13] A total of $\frac{n}{2}+1$ ($\frac{n+1}{2}+1$) voters will be needed for an even (odd) number of vertices. As before, both cases can be settled by the same constrution. Behind the preference profile lies the following structure. Starting from a hypothetical almost regular ordering

$$l_1, Y_1, g_2 \,;\, l_3, Y_3, g_3 \,;\, \ldots \,;\, l_{n-2}, Y_{n-2}, g_{n-2} \,;\, l_{n-1}, Y_{n-1}, g_n \,;\, l_2, Y_2, g_{n-1} \,;\, l_n, Y_n, g_1$$

[13]Note that Y_{n-2}, Y_{n-1} and Y_n are empty sets.

we generate column $k = 1, \ldots, \frac{n}{2}$ by exchanging edges l_{2k-1} and l_{2k-2} and sets Y_{2k-1} and Y_{2k-2} by edges l_2, l_n and sets Y_2 and Y_n (slightly abusing notation with $l_0 = l_n$). In addition we exchange in two rankings (columns 3 and 4) g_2 with g_n.

Starting from a global perspective, all edges in $G - \{g_1, g_{n-1}\}$ are approved by every individual. All other edges are not approved by at least one individual. Hence the global tour is $T_g = G$.

1	2	3	4	$\ldots$	$\frac{n}{2}$	$\frac{n}{2}+1$
l_1	l_1	l_1	l_1	$\ldots$	l_1	l_1
Y_1	Y_1	Y_1	Y_1	$\ldots$	Y_1	g_1
g_{n-1}	l_n	l_n	l_n	$\ldots$	l_n	l_n
l_n	g_{n-1}	g_{n-1}	g_{n-1}	$\ldots$	g_{n-1}	g_{n-1}
g_1	g_1	g_1	g_1	$\ldots$	g_1	Y_1

Table 19: AV - restricted profile at vertex 1.

Determining the local tour the feasible edges at vertex 1 are $E_1 = \{l_1, l_n, g_1, g_{n-1}\} \cup Y_1$. The reduced preference profile $P|E_1$ is given in Table 19. Obviously edge l_1 is the only edge to be approved by all voters and therefore the group moves along l_1 to vertex 2. There, the set of feasible edges reduces to $E_2 = \{l_2, g_2, g_n\} \cup Y_2$ and the preference profile, $P|E_2$ is given as in Table 20.

1	2	3	4	5	$\ldots$	$\frac{n}{2}$	$\frac{n}{2}+1$
l_2	g_2	g_n	g_n	g_2	$\ldots$	g_2	l_2
Y_2	g_n	l_2	l_2	l_2	$\ldots$	l_2	g_2
g_2	l_2	Y_2	Y_2	Y_2	$\ldots$	Y_2	g_n
g_n	Y_2	g_2	g_2	g_n	$\ldots$	g_n	Y_2

Table 20: AV - restricted profile at vertex 2.

At vertex 2, edge l_2 is approved by all but one voter whereas all other edges "loose" at least two approvals. Hence, the group moves along l_2 to vertex 3 where the set of feasible edges is $E_3 = \{l_3, g_3, Y_3\}$. As can be seen directly from Table 18 in this situation l_3 scores $n/2$ approvals, g_3 only 2 and each of the edges in Y_3 gets $n/2 - 1$ approvals. This scenario is replicated for $i > 3$ resulting in $T_\ell = L$. That T_ℓ pairwise dominates T_g can be checked in Table 18. ■

3.3 Other Voting Rules

A further investigation of two other well known voting rules, namely the Plurality Rule and the Simple Majority Rule, yields - in addition to some similar results as before - also different results which could be attributed to the particular way in which the individual preference information is used. In particular, using Plurality rule the global tour can never be pairwise dominated by any other tour, and for simple majority rule the global tour and the local tour can never be disjoint.

Proposition 6 *Using the Plurality Rule, there exist preference profiles such that the local tour is pairwise dominated by the global tour.*

Proof. Consider again Table 4. It is easy to see that all edges in $G - \{g_n\}$ get exactly one vote whereas all other edges get no votes. This, however, determines the global tour as $T_g = G$. Concerning the local tour we use Tables 5 and 6 and the same argumentation as in previous proofs to show that $T_\ell = L$. From Table 4 we observe that the local tour is pairwise dominated by the global tour. ■

Proposition 7 *Using the Plurality Rule, the global tour can never be pairwise dominated by any other tour.*

Proof. If a tour T is pairwise dominated, there must be $T \cap S_i^t = \emptyset, \forall\, i \in I$. Hence, all edges of such a tour would receive 0 votes by the Plurality Rule, which can not be the case for the tour determined by the global greedy rule. ■

As pointed out before the Simple Majority Rule does not necessarily yield a ranking of the edges. In fact it may induce cycles and thus prohibit the application of any iterative procedure to compute a tour. Technically, we resolve this issue by assigning the same rank to all edges in a cycle.[14]

In contrast to the Borda rule and Approval voting, by using Simple Majority Rule neither the local nor the global greedy rule can ever pairwise dominate each other. Intuitively, this is due to the fact that every global decision is based on the same pairwise comparisons as a local decision.

Proposition 8 *Using the Simple Majority Rule, the tours determined by the global greedy rule and by the local greedy rule can not be disjoint.*

Proof. Consider the two edges chosen as first and second edges by the global greedy rule. These are also the two best edges from the total ordering, since the selection of the first edge does not rule out any other edge from selection (which is not always true for the selection of the third edge, where a cycle may arise).

After passing through $|V| - 2$ vertices $V \setminus \{u, v\}$, the local greedy rule still has to decide between two edges, namely which of the remaining two vertices u and v should be chosen first. Hence, there is only the single edge $\{u, v\}$ which was never before part of any decision by the local greedy rule. All other edges will be taken into account at some point during the construction of the tour. In particular, if an edge $\{r, s\}$ appears for the first time in a decision step at vertex r, it is guaranteed that s was not yet chosen and hence this edge is eligible in the decision.

Therefore, at least one of the two best edges also takes part in a decision by the local greedy rule. By definition of the Simple Majority Rule, such an edge would win this vote and hence be part of both tours. ■

It should be noted that this statement holds only if the selection of the global greedy rule is unambiguous. The presence of a large set of edges with identical rank may leave open the possibility to construct a tour from these edges which is disjoint from the local tour.

[14] As there is a certain symmetry in the profile of our construction, most so-called simple majority extensions such as the Copeland rule, the Kemeny rule or Black's rule would yield this result. See Fishburn [6].

Corollary 1 *(1) Using Simple Majority Rule, the global tour can never be pairwise dominated by the local tour.*

(2) Using Simple Majority Rule, the local tour can never be pairwise dominated by the global tour.

Proof. Follows immediately from Proposition 8. ■

It is easy to see that the statements of Proposition 8 and Corollary 1 hold for every voting rule that satisfies the property of independence of irrelevant alternatives [19], i.e. in which the restriction to a subset of edges preserves the ordering of alternatives given by the global voting rule.

For the sake of completeness we will show that also for SMR each of the two greedy versions can be pairwise dominated. It follows trivially from Corollary 1 that this can not be a tour determined by the other greedy rule.

Proposition 9 *Using the Simple Majority Rule, for any graph V such that $V \geq 5$, there exist preference profiles such that the local tour is pairwise dominated.*

Proof.

profile			profile at vertex 1			profile at vertex 2		
1	2	3	1	2	3	1	2	3
g_1	g_2	g_3	g_1	l_1	l_1	l_2	g_2	l_2
l_1	l_1	l_1	l_1	g_1	g_1	g_2	l_2	g_2
g_{n-1}	g_1	g_1	g_{n-1}	g_{n-1}	g_{n-1}	g_n	g_n	g_n
l_2	l_2	l_2	l_n	l_n	l_n	X_2	X_2	X_2
g_2	g_{n-1}	g_2	X_1	X_1	X_1			
l_3	l_3	l_3						
g_3	g_3	g_{n-1}						
l_4	l_4	l_4						
g_4	g_4	g_4						
⋮	⋮	⋮						
l_{n-2}	l_{n-2}	l_{n-2}						
g_{n-2}	g_{n-2}	g_{n-2}						
l_{n-1}	l_{n-1}	l_{n-1}						
g_n	g_n	g_n						
l_n	l_n	l_n						
X	X	X						

Table 21: SMR - local tour pairwise dominated.

Consider as before a graph with an odd number of vertices as shown in Figure 1 and the individual preferences of 3 voters given in Table 21. All edges and edges sets are as previously defined. As can be seen from the restricted profiles for the local decisions at vertices 1 and 2 also given in Table 21, the local greedy rule starts the tour along edges l_1 and l_2. All further decisions between l_i and g_i, $i \geq 3$, are made in favour of the former leading to the solution of $T_\ell = L$ which is pairwise dominated by the tour

$T = G$. Note that the example holds also for $n = 5$, where T_ℓ is pairwise dominated by its complement. The case of an even number of vertices can be settled easily along the same lines as in previous examples. ■

The same statement can be shown to hold for the global greedy rule.

Proposition 10 *Using the Simple Majority Rule, for any graph V such that $V \geq 5$, there exist preference profiles such that the global tour is pairwise dominated.*

Proof.

1	2	3	edge ranking
l_2	l_6	l_1	g_1
l_3	l_7	l_{11}	g_{10}
g_1	g_{10}	g_1	l_1
g_{10}	g_1	g_{10}	l_{11}
l_1	l_{11}	l_4	g_3
l_{11}	l_1	l_5	g_5
g_3	g_5	g_3	l_4
g_5	g_3	g_5	l_5
l_4	l_5	l_8	g_7
l_5	l_4	l_9	g_9
g_7	g_9	g_7	l_8
g_9	g_7	g_9	l_9
l_8	l_9	$\vdots$	$\vdots$
l_9	l_8		X
$\vdots$	$\vdots$		
X	X	X	

Table 22: Example: $|V| = 11$: SMR - global tour pairwise dominated. The ranking of the remaining edges is irrelevant for the selection of the global tour as long as X is ranked below G.

1	2	3	ordering of edges
		l_{4k}	g_{4k-1}
		l_{4k+1}	g_{4k+1}
g_{4k-1}	g_{4k+1}	g_{4k-1}	l_{4k}
g_{4k+1}	g_{4k-1}	g_{4k+1}	l_{4k+1}
l_{4k}	l_{4k+1}		
l_{4k+1}	l_{4k}		

Table 23: Example: Canonical building block for $k = 1, 2, \ldots$, all edges emanating from vertex $4k + 1$.

The construction of our example is based on considering iteratively vertices 1, 5, 9 and onwards with a stepwidth of 4. For each such vertex $4k+1$, $k \geq 1$, SMR prefers the two emanating edges g_{4k-1} leading to vertex $4k-1$ and g_{4k+1} leading to vertex $4k+3$. Hence, vertex $4k+1$ is joined to two selected edges and all other adjacent edges such as l_{4k} and l_{4k+1} can not be selected anymore for a tour. This results in $T_g = G$ which is pairwise dominated by L. The canonical building block to reach this situation is given in Table 23. An example with $|V| = 11$ in Table 22 illustrates how this building block is used and how the special case of vertex 1 is dealt with. We refrain from going into the details of elaborating all cases of $|V| \neq 4k-1$ which require minor adaptations for vertices around vertex $|V|$. For sake of completeness we explicitly state in Table 24 the special case of $|V| = 5$ where T_g is again pairwise dominated by its complement. ■

1	2	3	edge ranking
l_3	l_4	l_2	g_1
g_1	g_4	g_1	g_4
l_1	l_5	l_5	l_5
g_4	g_1	g_4	l_1
l_5	l_1	l_1	g_3
g_3	g_3	g_3	l_2
l_2	l_2	l_3	l_3
l_4	l_3	l_4	l_4
g_2	g_2	g_2	g_2
g_5	g_5	g_5	g_5

Table 24: Example: $|V| = 5$: Global Greedy for Simple Majority Rule (SMR) pairwise dominated by its complement (in bold).

4 Conclusion

In this paper we have introduced and analyzed two algorithms from classical optimization to solve the traveling group problem. We have shown which problems can arise in such group decision situations and that neither of the two algorithms has always a clear advantage over the other for certain reasonable voting rules. Although the results in this paper are - in a certain sense - negative , this paper can only be seen as a first step into the analysis of group decisions in such a framework. Many - probably highly relevant - issues have not been discussed. Those include e.g. the importance of the starting vertex in our framework, the analysis of possible dominance between different voting rules, or the use of different structures such as that of spanning trees.

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