ARROVIAN IMPOSSIBILITIES IN AGGREGATING PREFERENCES OVER SETS

by

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February 2006

ABSTRACT

Given a society confronting a set of alternatives $\mathbf{A}$, we consider the aggregation of individual preferences over the power set $\underline{\mathbf{A}}$ of $\mathbf{A}$ into a social preference over $\underline{\mathbf{A}}$. In case we allow individuals to have any complete and transitive preference over $\underline{\mathbf{A}}$, Arrow's impossibility theorem naturally applies. However, the Arrovian impossibility prevails, even when the set of admissible preferences over $\underline{\mathbf{A}}$ is severely restricted by strong axioms that relate preferences over $\underline{\mathbf{A}}$ to preferences over $\mathbf{A}$. In fact, we identify a very narrow domain of lexicographic orderings over $\underline{\mathbf{A}}$ which exhibits the Arrovian impossibility in all of its superdomains. As the lexicographic extension we use is compatible with almost all standard extension axioms, we interpret our results as the strong prevalence of Arrow's impossibility theorem in aggregating preferences over sets.

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1. INTRODUCTION

Given a society confronting a set of alternatives $\mathbf{A}$, we consider the aggregation of individual preferences over the power set $\underline{\mathbf{A}}$ of $\mathbf{A}$ into a social preference over $\underline{\mathbf{A}}$. In case we allow individuals to have any complete and transitive preference over $\underline{\mathbf{A}}$, Arrow's impossibility theorem naturally applies. However, the Arrovian impossibility prevails, even when the set of admissible preferences over $\underline{\mathbf{A}}$ is severely restricted by strong axioms that relate preferences over $\underline{\mathbf{A}}$ to preferences over $\mathbf{A}$. In fact, we identify a very narrow domain of lexicographic orderings over $\underline{\mathbf{A}}$ which exhibits the Arrovian impossibility in all of its superdomains. As the lexicographic extension we use is compatible with almost all standard extension axioms, we interpret our results as the strong prevalence of Arrow's impossibility theorem in aggregating preferences over sets.

To be completed

2. BASIC NOTIONS

Taking any integer $n \geq 2$, we fix a society $\mathbf{N} = \{1,.., n\}$ confronting a non-empty finite set of alternatives $\mathbf{A}$ with $\#\mathbf{A} = m \geq 3$. Writing $\underline{\mathbf{A}} = 2^{\mathbf{A}} \setminus \{\emptyset\}$ for the set of all non-empty subsets of $\mathbf{A}$, we let $\Re$ stand for the set of all complete and transitive binary relations over $\underline{\mathbf{A}}$. Every $i \in \mathbf{N}$ is assumed to have a *preference* $R_i \in \Re$ over $\underline{\mathbf{A}}$.[1] A *preference profile* (over $\underline{\mathbf{A}}$) is an n-tuple $\underline{R} = (R_1,\ldots, R_n) \in \Re^{\mathbf{N}}$ of individual preferences.

Given any non-empty $D \subseteq \Re$, we define a *social welfare function* (SWF) over D as a mapping $\alpha: D^{\mathbf{N}} \rightarrow \Re$.[2] For any $\underline{R} \in D^{\mathbf{N}}$, we let $\alpha^*(\underline{R})$ stand for the strict counterpart of $\alpha(\underline{R})$.[3] A SWF $\alpha: D^{\mathbf{N}} \rightarrow \Re$ is said to be *Pareto optimal* (PO) iff given any $X \in \underline{\mathbf{A}}$, any $Y \in \underline{\mathbf{A}} \setminus \{X\}$ and any $\underline{R} \in D^{\mathbf{N}}$ with $X \, P_i \, Y$ for all $i \in \mathbf{N}$, we have $X \, \alpha^*(\underline{R}) \, Y$. We say that $\alpha: D^{\mathbf{N}} \rightarrow \Re$ is *independent of irrelevant alternatives* (IIA) iff given any $X \in \underline{\mathbf{A}}$, any $Y \in \underline{\mathbf{A}} \setminus \{X\}$ and any $\underline{R}$,

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[1] For any $X, Y \in \underline{\mathbf{A}}$, we interpret $X \, R_i \, Y$ as "X being at least as good as Y in view of individual i". We write $X \, P_i \, Y$ whenever we have $X \, R_i \, Y$ but not $Y \, R_i \, X$. We write $X \, I_i \, Y$ whenever $X \, R_i \, Y$ and $Y \, R_i \, X$ both hold.

[2] Note that we consider aggregation rules whose domains are Cartesian products of some $D \subseteq \Re$.

$\underline{R}' \in D^{\mathbf{N}}$ with $X\, R_i\, Y \Leftrightarrow X\, R_i'\, Y$ for all $i \in \mathbf{N}$, we have $X\, \underset{.}{\alpha}(\underline{R})\, Y \Leftrightarrow X\, \underset{.}{\alpha}(\underline{R}')\, Y$. We say that $\alpha: D^{\mathbf{N}} \rightarrow \Re$ is *dictatorial* iff there exists $d \in \mathbf{N}$ such that given any $\underline{R} \in D^{\mathbf{N}}$, any $X \in \underline{\mathbf{A}}$ and any $Y \in \underline{\mathbf{A}} \setminus \{X\}$, we have $X\, P_d\, Y \Rightarrow X\, \underset{.}{\alpha}^*(\underline{R})\, Y$. We qualify α as *non-dictatorial* (ND) whenever it is not dictatorial. Finally we say that $\alpha: D^{\mathbf{N}} \rightarrow \Re$ is *quasi-dictatorial* iff there exists $X \in \underline{\mathbf{A}}$, $d \in \mathbf{N}$ such that given any $\underline{R} \in D^{\mathbf{N}}$, any $Y \in \underline{\mathbf{A}} \setminus \{X\}$ and any $Z \in \underline{\mathbf{A}} \setminus \{X, Y\}$, we have $Y\, P_d\, Z \Rightarrow Y\, \underset{.}{\alpha}^*(\underline{R})\, Z$.

We qualify a non-empty domain $D \subseteq \Re$ as dictatorial (resp. quasi dictatorial) iff every PO and IIA SWF defined over D is dictatorial (resp. quasi dictatorial). An immediate consequence of Arrow's impossibility theorem is that the full domain $\Re$ is dictatorial. We ask whether it is possible to escape the Arrovian impossibility by natural restrictions of $\Re$ through axioms which extend preference orderings over alternatives to sets of alternatives.

We let Π stand for the set of all complete, transitive and antisymmetric binary relations over **A**. Every $i \in \mathbf{N}$ is assumed to have a *preference* $\rho_i \in \Pi$ over **A**.[4] A *preference profile* (over **A**) is an n-tuple $\underline{\rho} = (\rho_1, ..., \rho_n) \in \Pi^{\mathbf{N}}$ of individual preferences.

Now, given any $\rho \in \Pi$, we define the *leximax* extension of ρ is the ordering $\lambda^+(\rho)$ over $\underline{\mathbf{A}}$ as follows: Take any two distinct $X, Y \in \underline{\mathbf{A}}$. First consider the case where $\#X = \#Y = k$ for some $k \in \{1,\ldots,m\text{-}1\}$. Let, without loss of generality, $X = \{x_1,..., x_k\}$ and $Y = \{y_1,..., y_k\}$ such that $x_j\, \rho\, x_{j+1}$ and $y_j\, \rho\, y_{j+1}$ for all $j \in \{1,\ldots,k\text{-}1\}$. We have $X\, \lambda^+(\rho)\, Y$ if and only if $x_h\, \rho\, y_h$ for the smallest $h \in \{1,\ldots,k\}$ such that $x_h \neq y_h$. Now consider the case where $\#X \neq \#Y$. Let, without loss of generality, $X = \{x_1,..., x_{\#X}\}$ and $Y = \{y_1,..., y_{\#Y}\}$ such that $x_j\, \rho\, x_{j+1}$ for all $j \in \{1,\ldots, \#X\text{-}1\}$and $y_j\, \rho\, y_{j+1}$ for all $j \in \{1,\ldots, \#Y\text{-}1\}$. We have either $x_h = y_h$ for all $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ or there exists some $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case $X\, \lambda^+(\rho)\, Y$ if and only if $\#X < \#Y$. For the second case, $X\, \lambda^+(\rho)\, Y$ if and only if $x_h\, \rho\, y_h$ for the smallest $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.

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[3] So for all $X, Y \in \underline{\mathbf{A}}$, we have $X\, \underset{.}{\alpha}^*(R)\, Y$ iff $X\, \underset{.}{\alpha}(R)\, Y$ but not $Y\, \underset{.}{\alpha}(R)\, X$.

[4] For any $x, y \in \mathbf{A}$, we interpret $x\, \rho_i\, y$ as "x being at least as good as y in view of individual i". As ρ_i is antisymmetric, for any distinct $x, y \in \mathbf{A}$ we have $x\, \rho_i\, y \Rightarrow$ not $y\, \rho_i\, x$. In other words, for distinct alternatives, $x\, \rho_i\, y$ means "individual i prefers x to y".

So under the leximax extension, an agent orders two distinct sets according to their best elements. If these are the same, then the ordering is made according to their second best elements, etc. The elements according to which the sets are compared will disagree at some step – except possibly when one set is a subset of the other, in which case the smaller set is preferred.[5]

The concept of a *leximin* extension is similarly defined while it is based on ordering two sets according to a lexicographic comparison of their worst elements. Again the elements according to which the sets are compared will disagree at some step – except possibly when one set is a subset of the other, in which case the larger set is preferred. To say this formally, given any $\rho \in \Pi$, the *leximin* extension of ρ is the ordering $\lambda^{-}(\rho)$ over $\underline{\mathbf{A}}$ defined as follows: Take any two distinct $X, Y \in \underline{\mathbf{A}}$. First consider the case where $\#X = \#Y = k$ for some $k \in \{1,\ldots,m-1\}$. Let, without loss of generality, $X = \{x_1,\ldots, x_k\}$ and $Y = \{y_1,\ldots, y_k\}$ such that $x_j \ \rho \ x_{j+1}$ and $y_j \ \rho \ y_{j+1}$ for all $j \in \{1,\ldots,k-1\}$. We have $X \ \lambda^{-}(\rho) \ Y$ if and only if $x_h \ \rho \ y_h$ for the greatest $h \in \{1,\ldots,k\}$ such that $x_h \neq y_h$. Now consider the case where $\#X \neq \#Y$. Let, without loss of generality, $X = \{x_1,\ldots, x_{\#X}\}$ and $Y = \{y_1,\ldots, y_{\#Y}\}$ such that $x_{j+1} \ \rho \ x_j$ for all $j \in \{1,\ldots, \#X-1\}$and $y_{j+1} \ \rho \ y_j$ for all $j \in \{1,\ldots, \#Y-1\}$. We have either $x_h = y_h$ for all $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ or there exists some $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case $X \ \lambda^{-}(\rho) \ Y$ if and only if $\#X > \#Y$. For the second case, $X \ \lambda^{-}(\rho) \ Y$ if and only if $x_h \ \rho \ y_h$ for the smallest $h \in \{1,\ldots, \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.[6]

We first remark that at each $\rho \in \Pi$, the leximax and leximin extensions determine unique orderings $\lambda^{+}(\rho)$ and $\lambda^{-}(\rho)$ over $\underline{\mathbf{A}}$ which are complete, transitive and antisymmetric.[7] We write $D^{\lambda^+} = \{\lambda^{+}(\rho) : \ \rho \in \Pi\} \subset \Re$ and $D^{\lambda^-} = \{\lambda^{-}(\rho) : \ \rho \in \Pi\} \subset \Re$ for the sets of admissible orderings over $\underline{\mathbf{A}}$ respectively induced by the leximax and leximin extensions.

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[5] For example, given distinct $x, y, z \in \mathbf{A}$, the leximax extension of the ordering $x \ \rho \ y \ \rho \ z$ is $\{x\} \ \lambda^{+}(\rho) \ \{x, y\} \ \lambda^{+}(\rho) \ \{x, y, z\} \ \lambda^{+}(\rho) \ \{x, z\} \ \lambda^{+}(\rho) \ \{y\} \ \lambda^{+}(\rho) \ \{y, z\} \ \lambda^{+}(\rho) \ \{z\}$.

[6] For example, given distinct $x, y, z \in \mathbf{A}$, the leximin extension of the ordering $x \ \rho \ y \ \rho \ z$ is $\{x\} \ \lambda^{-}(\rho) \ \{x, y\} \ \lambda^{-}(\rho) \ \{y\} \ \lambda^{-}(\rho) \ \{x, z\} \ \lambda^{-}(\rho) \ \{x, y, z\} \ \lambda^{-}(\rho) \ \{y, z\} \ \lambda^{-}(\rho) \ \{z\}$.

[7] This is already shown by Kaymak and Sanver (2003) and we do not reproduce the proof here. However, we wish to underline that the antisymmetry of $\lambda^{+}(\rho)$ and $\lambda^{-}(\rho)$ is critical in our analysis which follows.

3. MAIN RESULTS

We collect in this section our main results on the (im)possibility of defining Arrovian SWFs over D^{λ^+} and/or D^{λ^-}. We give the proofs at the appendix.

Theorem 3.1: Both D^{λ^+} and D^{λ^-} are quasi-dictatorial. In fact, given any PO and IIA SWF α: $[D^{\lambda^+}]^{\mathbf{N}} \rightarrow \Re$, there exists $d \in \mathbf{N}$ such that at each $\underline{R} \in D^{\mathbf{N}}$, we have $X \; P_d \; Y \Rightarrow X \; \underset{\cdot}{\alpha}^*(\underline{R}) \; Y$ for all distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$. Similarly, given any PO and IIA SWF α: $[D^{\lambda^-}]^{\mathbf{N}} \rightarrow \Re$, there exists $d \in \mathbf{N}$ such that at each $\underline{R} \in D^{\mathbf{N}}$, we have $X \; P_d \; Y \Rightarrow X \; \underset{\cdot}{\alpha}^*(\underline{R}) \; Y$ for all distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$.

Proof: See Appendix A

Theorem 3.2: The domains D^{λ^+} and D^{λ^-} are not dictatorial.

Proof: See Appendix B.

Remark: As expected, Theorem 3.2 is proven by giving two examples of PO, IIA and non-dictatorial SWFs, one defined over D^{λ^+} and the other over D^{λ^-}. Nevertheless, we know by Theorem 3.1 that, both of these SWFs must exhibit an individual who is the dictator over all sets but **A**.

Theorem 3.3: The domain $D^{\lambda^+} \cup D^{\lambda^-}$ is superdictatorial.

Proof: See Appendix C.

CONCLUSION AND REFERENCES

To be completed

APPENDIX A

Theorem 3.1: Both D^{λ^+} and D^{λ^-} are quasi-dictatorial. In fact, given any PO and IIA SWF α: $[D^{\lambda^+}]^{\mathbf{N}} \rightarrow \Re$, there exists $d \in \mathbf{N}$ such that at each $\underline{R} \in D^{\mathbf{N}}$, we have $X \, P_d \, Y \Rightarrow X \, \alpha^*(\underline{R}) \, Y$ for all distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$. Similarly, given any PO and IIA SWF α: $[D^{\lambda^-}]^{\mathbf{N}} \rightarrow \Re$, there exists $d \in \mathbf{N}$ such that at each $\underline{R} \in D^{\mathbf{N}}$, we have $X \, P_d \, Y \Rightarrow X \, \alpha^*(\underline{R}) \, Y$ for all distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$.

Proof: The proof uses the standard "free triple" concept of the literature[8] and benefits from a graph theoretic technique introduced by Ozdemir and Sanver (2006).

By a pair (resp., triple) of sets we mean two (resp. three) distinct sets. A pair $X, Y \in \underline{\mathbf{A}}$ is said to be *non-trivial* in a domain $D \subseteq \Re$ if and only if there exist $P, P' \in D$ with $X \, P \, Y$ and $Y \, P \, X$. A triple $X, Y, Z \in \underline{\mathbf{A}}$ is said to be *free* in a domain $D \subseteq \Re$ if and only if given any distinct $B_1, B_2, B_3 \in \{X, Y, Z\}$, there exists $P \in D$ with $B_1 \, P \, B_2 \, P \, B_3$.

The *local dictatoriality graph* $G(D)$ of a domain D is a graph whose vertices are the non-trivial pairs in D. Any two vertices $\{X, Y\}$ and $\{Z, W\}$ are linked if and only if $\{X, Y\} \cup \{Z, W\}$ forms a free triple in D.[9] Ozdemir and Sanver (2006) show that every connected component[10] of $G(D)$ which contains more than one vertex is "locally dictatorial", i.e., given any PO and IIA SWF α: $D^{\mathbf{N}} \rightarrow \Re$ and any connected component G of $G(D)$ which contains more than one vertex, there exists $d \in \mathbf{N}$ such that for any $\underline{R} \in D^{\mathbf{N}}$ and any vertex $\{X, Y\}$ of G, we have $X \, P_d \, Y \Rightarrow X \, \alpha^*(\underline{R}) \, Y$.

We show the quasi-dictatoriality of D^{λ^+} by showing that its local dictatoriality graph $G(D^{\lambda^+})$ has a connected component G such that any $X \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ is a vertex of G.

(to be completed)

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[8] Typical examples of which can be found in Blau (1957), Arrow (1963), Kalai and Muller (1977), Kalai, Muller and Satterthwaite (1979) and Kelly (1994).

[9] Thus {x, y} and {z, w} are linked only if they have exactly one element in common.

APPENDIX B

Theorem 3.2: The domains D^{λ^+} and D^{λ^-} are not dictatorial.

Proof: We start by recalling that at each $\rho \in \Pi$, the leximax and leximin extensions determine unique orderings $\lambda^+(\rho)$ and $\lambda^-(\rho)$ over $\underline{\mathbf{A}}$ which are complete, transitive and antisymmetric.[11]

We state two lemmata about the properties of the leximax extension:

Lemma B.1: Given any $\rho \in \Pi$ and any distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ with $X\ \lambda^+(\rho)\ \mathbf{A}$ and $Y\ \lambda^+(\rho)\ \mathbf{A}$, we have

(i) $\#X \neq \#Y$

(ii) $X\ \lambda^+(\rho)\ Y \Leftrightarrow \#X < \#Y$.

Proof: Take any $\rho \in \Pi$ and any distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ with $X\ \lambda^+(\rho)\ \mathbf{A}$ and $Y\ \lambda^+(\rho)\ \mathbf{A}$. Write $\mathbf{A} = \{a_1,..., a_m\}$ such that $a_j\ \rho\ a_{j+1}$ for all $j \in \{1,\dots, m-1\}$. Remark that $X\ \lambda^+(\rho)\ \mathbf{A}$, implies $X = \{a_j \in \mathbf{A} : j \leq h \text{ for some } h \in \{1,\dots,m-1\}\}$. Similarly $Y\ \lambda^+(\rho)\ \mathbf{A}$ implies $Y = \{a_j \in \mathbf{A} : j \leq h' \text{ for some } h' \in \{1,\dots, m-1\}\}$. As X and Y are distinct, h and h' cannot be the same, which establishes $\#X \neq \#Y$, showing part (i) of the lemma. We now show part (ii). First let $\#X < \#Y$. Thus, $h < h'$, which, by the definition of the leximax extension, implies $X\ \lambda^+(\rho)\ Y$. Now let $\#X \geq \#Y$ which, by part (i) of this lemma, implies $\#Y < \#X$. Thus, $h' < h$, which, by the definition of the leximax extension, implies $Y\ \lambda^+(\rho)\ X$. Q.E.D.

Lemma B.2: Given any $\rho \in \Pi$ and any distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ with $\#X \geq \#Y$, we have $X\ \lambda^+(\rho)\ Y \Rightarrow \mathbf{A}\ \lambda^+(\rho)\ Y$.

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[10] A graph G is said to be connected if and only if given any two vertices v and v' of G, there is a sequence $v = v_1,..., v_s = v'$ of vertices of G such that v_i and v_{i+1} are linked for each $i = 1,..., s-1$. So by a connected component of G(D), we mean a maximal connected subgraph G of G(D), i.e., G is connected but no supergraph of G is so.

[11] See Footnote 7.

Proof: Take any $\rho \in \Pi$ and any distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ with $\#X \geq \#Y$ such that $X\ \lambda^+(\rho)\ Y$. Suppose for a contradiction that $Y\ \lambda^+(\rho)\ \mathbf{A}$. The transitivity of $\lambda^+(\rho)$ implies $X\ \lambda^+(\rho)\ \mathbf{A}$. So, by Lemma B.1, we have $Y\ \lambda^+(\rho)\ X$, giving the required contradiction. Q.E.D.

We now see that D^{λ^+} is not dictatorial. Consider the SWF α^+: $[D^{\lambda^+}]^{\mathbf{N}} \rightarrow \Re$ where for any $\underline{R} \in [D^{\lambda^+}]^{\mathbf{N}}$ and any distinct $X, Y \in \underline{\mathbf{A}}$ we have $X\ \alpha^+(\underline{R})\ Y \Leftrightarrow X\ P_1\ Y$ except when $X = \mathbf{A}$ and $\#Y = m\text{-}1$, in which case we have $Y\ \alpha^+(\underline{R})\ X \Leftrightarrow Y\ P_i\ X$ for all $i \in \mathbf{N}$. It is straightforward to check that α^+ satisfies PO and IIA. We establish the non-dictatoriality of D^{λ^+} by showing that at any $\underline{R} \in [D^{\lambda^+}]^{\mathbf{N}}$, we have $X\ \alpha^+(\underline{R})\ Y$ and $Y\ \alpha^+(\underline{R})\ Z \Rightarrow X\ \alpha^+(\underline{R})\ Z$ for all $X, Y, Z \in \underline{\mathbf{A}}$. So take any $\underline{R} \in [D^{\lambda^+}]^{\mathbf{N}}$ and any $X, Y, Z \in \underline{\mathbf{A}}$ with $X\ \alpha^+(\underline{R})\ Y$ and $Y\ \alpha^+(\underline{R})\ Z$. We will show $X\ \alpha^+(\underline{R})\ Z$ for the following 13 exhaustive cases:

CASE 1: $X, Y, Z \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$ and as $Y\ \alpha^+(\underline{R})\ Z$ we have $Y\ P_1\ Z$. The transitivity of P_1 implies $X\ P_1\ Z$ which, by the definition of α^+, gives $X\ \alpha^+(\underline{R})\ Z$.

CASE 2: $X = \mathbf{A}$ and $\#Y, \#Z < m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$ and as $Y\ \alpha^+(\underline{R})\ Z$ we have $Y\ P_1\ Z$. The transitivity of P_1 implies $X\ P_1\ Z$ which, by the definition of α^+, gives $X\ \alpha^+(\underline{R})\ Z$.

CASE 3: $X = \mathbf{A}$, $\#Y = m\text{-}1$ and $\#Z < m\text{-}1$. As $Y\ \alpha^+(\underline{R})\ Z$ we have $Y\ P_1\ Z$. Since $\#Y > \#Z$, by Lemma B.2, we have $X\ P_1\ Z$, hence, $X\ \alpha^+(\underline{R})\ Z$.

CASE 4: $X = \mathbf{A}$, $\#Y < m\text{-}1$ and $\#Z = m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$ and as $Y\ \alpha^+(\underline{R})\ Z$, we have $Y\ P_1\ Z$. The transitivity of P_1 implies $X\ P_1\ Z$. So, there exists $i \in \mathbf{N}$ with $X\ P_i\ Z$, which implies $X\ \alpha^+(\underline{R})\ Z$.

CASE 5: $X = \mathbf{A}$ and $\#Y = \#Z = m\text{-}1$. As $Y\ \alpha^+(\underline{R})\ Z$, we have $Y\ P_1\ Z$. Since $\#Y = \#Z$, by Lemma B.2, we have $X\ P_1\ Z$. So, there exists $i \in \mathbf{N}$ with $X\ P_i\ Z$, which implies $X\ \alpha^+(\underline{R})\ Z$.

CASE 6: $Y = \mathbf{A}$ and $\#X, \#Z < m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$ and as $Y\ \alpha^+(\underline{R})\ Z$ we have $Y\ P_1\ Z$. The transitivity of P_1 implies $X\ P_1\ Z$ which, by the definition of α^+, gives $X\ \alpha^+(\underline{R})\ Z$.

CASE 7: $Y = \mathbf{A}$, $\#X = m\text{-}1$ and $\#Z < m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$, we have $X\ P_i\ Y$ for all $i \in \mathbf{N}$, thus $X\ P_1\ Y$. As $Y\ \alpha^+(\underline{R})\ Z$, we have $Y\ P_1\ Z$. for all $i \in \mathbf{N}$, The transitivity of P_1 implies $X\ P_1\ Z$ which, by the definition of α^+, gives $X\ \alpha^+(\underline{R})\ Z$.

CASE 8: $Y = \mathbf{A}$, $\#X < m\text{-}1$ and $\#Z = m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$. In case $Y\ P_1\ Z$, we have $X\ P_1\ Z$, thus $X\ \alpha^+(\underline{R})\ Z$. In case $Z\ P_1\ Y$, as $\#X < \#Z$, Lemma B.1 establishes $X\ P_1\ Z$, thus $X\ \alpha^+(\underline{R})\ Z$.

CASE 9: $Y = \mathbf{A}$ and $\#X = \#Z = m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$, we have $X\ P_i\ Y$ for all $i \in \mathbf{N}$, thus $X\ P_1\ Y$. Note that $Z\ P_1\ Y$ cannot hold, as otherwise $\#X = \#Z$ would contradict Lemma B.1. So we have $Y\ P_1\ Z$, which, by transitivity of P_1 implies $X\ P_1\ Z$, thus $X\ \alpha^+(\underline{R})\ Z$.

CASE 10: $Z = \mathbf{A}$ and $\#X, \#Y < m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$ and as $Y\ \alpha^+(\underline{R})\ Z$, we have $Y\ P_1\ Z$. The transitivity of P_1 implies $X\ P_1\ Z$, thus $X\ \alpha^+(\underline{R})\ Z$.

CASE 11: $Z = \mathbf{A}$, $\#X < m\text{-}1$ and $\#Y = m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$ and as $Y\ \alpha^+(\underline{R})\ Z$, we have $Y\ P_i\ Z$ for all $i \in \mathbf{N}$, thus $Y\ P_1\ Z$. The transitivity of P_1 implies $X\ P_1\ Z$, hence $X\ \alpha^+(\underline{R})\ Z$.

CASE 12: $Z = \mathbf{A}$, $\#X = m\text{-}1$ and $\#Y < m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$ and as $Y\ \alpha^+(\underline{R})\ Z$, we have $Y\ P_1\ Z$. The transitivity of P_1 implies $X\ P_1\ Z$. As $X\ P_1\ Z$, $Y\ P_1\ Z$ and $\#X > \#Y$, by Lemma B.1, we have $Y\ P_1\ X$, contradicting the initial choice of X, Y and P_1.

CASE 13: $Z = \mathbf{A}$ and $\#X = \#Y = m\text{-}1$. As $X\ \alpha^+(\underline{R})\ Y$ we have $X\ P_1\ Y$ and as $Y\ \alpha^+(\underline{R})\ Z$, we have $Y\ P_i\ Z$ for all $i \in \mathbf{N}$, thus $Y\ P_1\ Z$. The transitivity of P_1 implies $X\ P_1\ Z$. But by Lemma B.1, $X\ P_1\ Z$, $Y\ P_1\ Z$ and $\#X = \#Y$ is not possible.

We now show the non-dictatoriality of D^{λ^-}. We first state two lemmata about the properties of the leximin extension:

Lemma B.3: Given any $\rho \in \Pi$ and any distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ with $\mathbf{A}\ \lambda^-(\rho)\ X$ and $\mathbf{A}\ \lambda^-(\rho)\ Y$, we have

(i) $\#X \neq \#Y$

(ii) $X\ \lambda^{-}(\rho)\ Y \Leftrightarrow \#X > \#Y$.

Proof: Take any $\rho \in \Pi$ and any distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ with $\mathbf{A}\ \lambda^{-}(\rho)\ X$ and $\mathbf{A}\ \lambda^{-}(\rho)\ Y$. Write $\mathbf{A} = \{a_1, \dots, a_m\}$ such that $a_j\ \rho\ a_{j+1}$ for all $j \in \{1, \dots, m-1\}$. Remark that $\mathbf{A}\ \lambda^{-}(\rho)\ X$ implies $X = \{a_j \in \mathbf{A} : j \geq h \text{ for some } h \in \{2, \dots, m\}\}$. Similarly $\mathbf{A}\ \lambda^{-}(\rho)\ Y$ implies $Y = \{a_j \in \mathbf{A} : j \geq h' \text{ for some } h \in \{2, \dots, m\}\}$. As X and Y are distinct, h and h' differ, which establishes $\#X \neq \#Y$, showing part (i) of the lemma. We now show part (ii). First let $\#X > \#Y$. Thus $h < h'$, which, by the definition of the leximin extension, implies $X\ \lambda^{-}(\rho)\ Y$. Now let $\#Y \geq \#X$ which, by part (i) of this lemma, implies $\#Y > \#X$. Thus $h' < h$, which, by the definition of the leximin extension, implies $Y\ \lambda^{-}(\rho)\ X$. Q.E.D.

Lemma B.4: Given any $\rho \in \Pi$ and any distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ with $\#X \geq \#Y$, we have $Y\ \lambda^{-}(\rho)\ X \Rightarrow Y\ \lambda^{-}(\rho)\ \mathbf{A}$.

Proof: Take any $\rho \in \Pi$ and any distinct $X, Y \in \underline{\mathbf{A}} \setminus \{\mathbf{A}\}$ with $\#X \geq \#Y$ such that $Y\ \lambda^{-}(\rho)\ X$. Suppose for a contradiction that $\mathbf{A}\ \lambda^{-}(\rho)\ Y$. The transitivity of $\lambda^{-}(\rho)$ implies $\mathbf{A}\ \lambda^{+}(\rho)\ X$. So, by Lemma B.3, we have $X\ \lambda^{+}(\rho)\ Y$, giving the required contradiction. Q.E.D.

To see the non-dictatoriality of $D^{\lambda^{-}}$, consider the SWF $\alpha^{-}: [D^{\lambda^{-}}]^{\mathbf{N}} \rightarrow \Re$ where for any $\underline{R} \in [D^{\lambda^{-}}]^{\mathbf{N}}$ and any distinct $X, Y \in \underline{\mathbf{A}}$ we have $X\ \alpha^{-}(\underline{R})\ Y \Leftrightarrow X\ P_1\ Y$ except when $X = \mathbf{A}$ and $\#Y = m-1$, in which case we have $X\ \alpha^{-}(\underline{R})\ Y \Leftrightarrow X\ P_i\ Y$ for all $i \in \mathbf{N}$. It is straightforward to check that α^{-} satisfies PO and IIA. So the proof is completed by showing that at any $\underline{R} \in [D^{\lambda^{-}}]^{\mathbf{N}}$, we have $X\ \alpha^{-}(\underline{R})\ Y$ and $Y\ \alpha^{-}(\underline{R})\ Z \Rightarrow X\ \alpha^{-}(\underline{R})\ Z$ for all $X, Y, Z \in \underline{\mathbf{A}}$. This can be seen exactly in the same way as we have done for α^{+}, with the sole change of replacing Lemma B1 with Lemma B3 and Lemma B2 with Lemma B4.

APPENDIX C

(to be completed)